MODAL ANALYSIS

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Glossary

DOF: Degree of freedom.
FRF: Frequency response function.
GTLS: Generalized total least squares.
IQML: Iterative quadratic maximum likelihood.
IRF: Impulse response function.
LS: Least squares.
LSCE: Least squares complex exponential.
MDOF: Multiple-degree-of-freedom.
ML: Maximum likelihood.
MIMO: Multiple-input-multiple-output
TLS: Total least squares.
Summary

In this contribution the applicability of frequency-domain estimators in the field of modal analysis will be illustrated. The basics of vibration and modal analysis are briefly summarized. In modal analysis, mechanical systems with a few inputs and hundreds of outputs have to be identified. This requires adapted frequency-domain estimators designed to handle large amount of data in a reasonable amount of time. A practical example will be given and finally the conclusions will be drawn.

1. Introduction

It is well known that (mechanical) structures can resonate, i.e. that small forces can result in important deformation, and possibly, damage can be induced in the structure.

The Tacoma Narrows bridge disaster (Figure 1) is a typical example of this. On November 7, 1940, the Tacoma Narrows suspension bridge collapsed due to wind-induced vibration (i.e. flutter). Situated on the Tacoma Narrows in Puget Sound, near the city of Tacoma, Washington, the bridge had only been open for traffic a few months.

Wings of airplanes can be subjected to similar flutter phenomena during flight. Before an airplane is released, flight flutter tests have to be performed to detect possible onset of flutter. The classical flight flutter testing approach is to expand the flight envelope of a airplane by performing a vibration test at constant flight conditions, curve-fit the data to estimate the resonance frequencies and damping ratios, and then to plot these frequencies and damping estimates against flight speed or Mach number. The damping values are then extrapolated in order to determine whether it is save to proceed to the next flight test point. Flutter will occur when one of the damping values tends to become negative. Before starting the flight tests, ground vibration tests as well as numerical simulations and wind tunnel tests (see Figure 2) are used to get some prior insight into the problem.
The majority of structures can be made to resonate, i.e. to vibrate with excessive oscillatory motion. Resonant vibration is mainly caused by an interaction between the inertial and elastic properties of the materials within a structure. Resonance is often the cause of, or at least a contributing factor to many of the vibration and noise related problems that occur in structures and operating machinery. To better understand any structural vibration problem, the resonant frequencies of a structure need to be identified and quantified. Today, modal analysis has become a widespread means of finding the modes of vibration of a machine or structure (Figure 3). In every development of a new or improved mechanical product, structural dynamics testing on product prototypes is used to assess its real dynamic behavior.

2. The “Modal” Model

Modes are inherent properties of a structure, and are determined by the material properties (mass, damping, and stiffness), and boundary conditions of the structure. Each mode is defined by a natural (modal or resonant) frequency, modal damping, and a mode shape (i.e. the so-called “modal parameters”). If either the material properties or the boundary conditions of a structure change, its modes will change. For instance, if mass is added to a structure, it will vibrate differently. To understand this, we will make use of the concept of single and multiple-degree-of-freedom systems.
2.1 Single Degree of Freedom

A single-degree-of-freedom (SDOF) system (see Figure 4 where the mass $m$ can only move along the vertical x-axis) is described by the following equation

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t)$$

(1)

with $m$ the mass, $c$ the damping coefficient, and $k$ the stiffness. This equation states that the sum of all forces acting on the mass $m$ should be equal to zero with $f(t)$ an externally applied force, $-m\ddot{x}(t)$ the inertial force, $-c\dot{x}(t)$ the (viscous) damping force, and $-kx(t)$ the restoring force. The variable $x(t)$ stands for the position of the mass $m$ with respect to its equilibrium point, i.e. the position of the mass when $f(t) \equiv 0$. Transforming (1) to the Laplace domain (assuming zero initial conditions) yields

$$Z(s)X(s) = F(s)$$

(2)

with $Z(s)$ the dynamic stiffness

$$Z(s) = ms^2 + cs + k$$

(3)

The transfer function $H(s)$ between displacement and force, $X(s) = H(s)F(s)$, equals the inverse of the dynamic stiffness

$$H(s) = \frac{1}{ms^2 + cs + k}$$

(4)

The roots of the denominator of the transfer function, i.e. $d(s) = ms^2 + cs + k$, are the poles of the system. In mechanical structures, the damping coefficient $c$ is usually very small resulting in a complex conjugate pole pair

$$\lambda = -\sigma \pm i\omega_d$$

(5)

with $\omega_d = \omega_n / 2\pi$ the damped natural frequency,

$$\omega_n = \sqrt{k/m}$$

the (undamped) natural frequency where $\omega_n = \sqrt{k/m} = |\lambda|$, and

$$\zeta = c/(2m\omega_n) = \sigma/|\lambda|$$

the damping ratio ($f_d = f_n\sqrt{1 - \zeta^2}$).

If, for instance, a mass $\Delta m$ is added to the original mass $m$ of the structure, its natural frequency decreases to $\omega_n = \sqrt{k/(m + \Delta m)}$. If $c = 0$, the system is not damped and the poles becomes purely imaginary, $\lambda = \pm i\omega_n$. 

![Figure 4: SDOF system.](image-url)
The Frequency Response Function (FRF), denoted by $H(\omega)$, is obtained by replacing the Laplace variable $s$ in (4) by $i\omega$ resulting in

$$H(\omega) = \frac{1}{-m\omega^2 + ic\omega + k} = \frac{1}{(k - m\omega^2) + ic\omega}$$

(6)

Clearly, if $c = 0$, then $H(\omega)$ goes to infinity for $\omega \to \omega_n = \sqrt{k/m}$ (see Figure 4).

Although very few practical structures could realistically be modeled by a single-degree-of-freedom (SDOF) system, the properties of such a system are important because those of a more complex multiple-degree-of-freedom (MDOF) system can always be represented as the linear superposition of a number of SDOF characteristics (when the system is linear time-invariant).

### 2.2 Multiple Degree of Freedom

Multiple-degree-of-freedom (MDOF) systems are described by the following equation

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = f(t)$$

(7)

In Figure 5, the different matrices are defined for a 2-DOF system with both DOF along the vertical x-axis.

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad f(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}, \quad K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}, \quad C = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}$$

Figure 5: 2-DOF system.

Transforming (7) to the Laplace domain (assuming zero initial conditions) yields

$$Z(s)X(s) = F(s)$$

(8)

with $Z(s)$ the dynamic stiffness matrix

$$Z(s) = Ms^2 + Cs + K$$

(9)

The transfer function matrix $H(s)$ between displacement and force vectors, $X(s) = H(s)F(s)$, equals the inverse of the dynamic stiffness matrix

$$H(s) = [Ms^2 + Cs + K]^{-1} = \frac{N(s)}{d(s)}$$

(10)

with the numerator polynomial matrix $N(s)$ given by

$$N(s) = \text{adj}(Ms^2 + Cs + K)$$

(11)

and the common-denominator polynomial $d(s)$, also known as the characteristic polynomial,

$$d(s) = \det(Ms^2 + Cs + K)$$

(12)
When the damping is small, the roots of the characteristic polynomial $d(s)$ are complex conjugate pole pairs, $\lambda_m$ and $\lambda_m^*$, $m = 1, \ldots, N_m$, with $N_m$ the number of modes of the system. The transfer function can be rewritten in a pole-residue form, i.e. the so-called “modal” model (assuming all poles have multiplicity one)

$$H(s) = \sum_{m=1}^{N_m} \frac{R_m}{s - \lambda_m} + \frac{R_m^*}{s - \lambda_m^*}$$

(13)

The residue matrices $R_m$, $m = 1, \ldots, N_m$, are defined by

$$R_m = \lim_{s \to \lambda_m} H(s)(s - \lambda_m)$$

(14)

It can be shown that the matrix $R_m$ is of rank one meaning that $R_m$ can be decomposed as

$$R_m = \psi_m \psi_m^T = \begin{bmatrix} \psi_m(1) \\ \psi_m(2) \\ \vdots \\ \psi_m(N_m) \end{bmatrix} \begin{bmatrix} \psi_m(1) & \psi_m(2) & \cdots & \psi_m(N_m) \end{bmatrix}$$

(15)

with $\psi_m$ a vector representing the “mode shape” of mode $m$. From equation (13), one concludes that the transfer function matrix of a linear time-invariant MDOF system with $N_m$ DOFs is the sum of $N_m$ SDOF transfer functions (“modal superposition”) and that the full transfer function matrix is completely characterized by the modal parameters, i.e. the poles $\lambda_m = -\sigma_m \pm i\omega_{d,m}$ and the mode shape vectors $\psi_m$, $m = 1, \ldots, N_m$.

Taking the inverse Laplace transform of (13) gives the Impulse Response Function (IRF)

$$h(t) = \sum_{m=1}^{N_m} R_m e^{\lambda_m t} + R_m^* e^{\lambda_m^* t}$$

(16)

which consists of a sum of complex exponential functions.

### 2.2.1 Mode Shapes and Operating Deflection Shapes

At or near the natural frequency of a mode, the overall vibration shape (“operating deflection shape”) of a structure will tend to be dominated by the mode shape of the resonance. Applying a harmonic force at one of the DOFs, say $r$, with angular frequency corresponding to the damped natural frequency of for instance the $n$-th mode, i.e. $s = i\omega_{d,n}$, results in a displacement vector that is approximately equal to

$$X(\omega_{d,n}) = \frac{\psi_n^{(r)} F_r(\omega_{d,n})}{\sigma_n}$$

(17)

The observed displacement at angular frequency $\omega_{d,n}$ is called an “operating deflection shapes” and is approximately proportional to the mode shape vector of mode $n$, i.e. $\psi_n$. When $\sigma_n$ is small, the proportionality is well satisfied. In reality, there will always be a (small) contribution of the other modes resulting in

$$X(\omega_{d,n}) = \frac{\psi_n^{(r)} F_r(\omega_{d,n})}{\sigma_n} + \sum_{m=1}^{N_m} \frac{\psi_m^{(r)} F_m(\omega_{d,n})}{\sigma_m + i(\omega_n - \omega_m)} + \sum_{m=1}^{N_m} \frac{\psi_m^{(r)} F_m(\omega_{d,n})}{\sigma_m + i(\omega_n + \omega_m)}$$

(18)
2.2.2 Observability and Controllability of Modes

Assuming, for example, that one force is applied in DOF 1 while the displacement is observed in DOF 2. In that case, the multiple-input-multiple-output (MIMO) transfer function matrix simplifies to the following single-input-single-output (SISO) transfer function

\[ H_{2,1}(s) = \sum_{m=1}^{N} \frac{\psi_m^*(2) \cdot \psi_m^*(1)}{s - \lambda_m} + \frac{\psi_m^*(2) \cdot \psi_m^*(1)}{s - \lambda_m} \]  (19)

If \( \psi_n(1) \neq 0 \) then mode \( n \) will only be “observed” in DOF 2 if \( \psi_n(2) \neq 0 \). If \( \psi_n(1) = 0 \) then it is clear that the terms corresponding to mode \( n \) will not appear in the sum, i.e. mode \( n \) cannot be excited (or “controlled”) by applying a force in DOF 1. The DOFs where a mode shape vector equals zero are called nodal points or nodes. In practice, this means that the force actuator should not be positioned in a nodal point of the modes of interest. To reduce the risk of missing modes, the number of excitation points can be increased. The same is true for the response measurements. The number of inputs (excitation points) is typically in the order of 1 to 10, while the number of outputs (response measurements) can reach more than 1000 points when using optical measurement equipment such as for instance a scanning laser Doppler vibrometer.

3. Frequency-Domain Identification of Modes

Typical for modal analysis is the very large number of outputs. This huge amount of data requires dedicated algorithms that balance between accuracy and memory/computing needs. In Section 3.1 such a ‘dedicated’ frequency-domain least-squares estimator will be presented. Based on these results, it is possible to implement more sophisticated identification methods (see \[ \text{6.43.8.2 Estimation with Known Noise Model} \] and \[ \text{6.43.8.4 Estimation with Unknown Noise Model} \]) such as for instance the frequency-domain Maximum Likelihood (ML) estimator (Section 3.2).

3.1 Least Squares Estimation

3.1.1 Common-Denominator Model

The relationship between output \( o \) (\( o = 1, \ldots, N_o \)) and input \( i \) (\( i = 1, \ldots, N_i \)) is modeled in the frequency domain by means of a common-denominator transfer function

\[ \hat{H}_k(\omega) = \frac{N_k(\omega)}{d(\omega)} \]  (20)

for \( k = 1, \ldots, N_oN_i \) (where \( k = (o-1)N_i + i \)) and with

\[ N_k(\omega) = \sum_{j=0}^{n} \Omega_j(\omega)B_{kj} \]  (21)

the numerator polynomial between output \( o \) and input \( i \) and

\[ d(\omega) = \sum_{j=0}^{n} \Omega_j(\omega)A_j \]  (22)

the common-denominator polynomial. The real-valued coefficients \( A_j \) and \( B_{kj} \) are the parameters to be estimated. Several choices are possible for the polynomial basis functions \( \Omega_j(\omega) \). For a discrete-time domain model, the functions \( \Omega_j(\omega) \) are usually given by \( \Omega_j(\omega) = \exp(-i\omega T_s \cdot j) \) (with \( T_s \) the sampling period) while for a continuous-time domain model \( \Omega_j(\omega) = (i\omega)^j \). The bad
numerical conditioning of the continuous-time domain approach can be improved by using for instance orthogonal Forsythe polynomials (at the expense of an increase of the computation time).

### 3.1.2 Linearity in the Parameters

In modal analysis, measurements of Frequency Response Functions (FRFs) are commonly used (see 6.43.8.1 Measurements of Frequency response functions). Replacing the model \( \hat{H}_k(\omega) \) in (20) by the measured FRFs \( H_k(\omega_j) \) for \( f=1,\ldots,N_f \) gives, after multiplication with the denominator polynomial,

\[
\sum_{j=0}^{n} \Omega_j(\omega_j)B_{kj} - \sum_{j=0}^{n} \Omega_j(\omega_j)H_k(\omega_j)A_j \approx 0
\]  

(23)

with \( k=1,\ldots,N_o, N_i \) and \( f=1,\ldots,N_f \). Because the equations (23) are “linear-in-the-parameters”, they can be reformulated as

\[
\begin{bmatrix}
X_1 & 0 & \cdots & 0 & Y_1 \\
0 & X_2 & 0 & \cdots & Y_2 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & X_{N_o N_i} & Y_{N_o N_i}
\end{bmatrix}
\begin{bmatrix}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_{N_o N_i}
\end{bmatrix}
= \begin{bmatrix}
A_0 \\
A_1 \\
\vdots \\
A_n
\end{bmatrix}
\]

(24)

with \( \beta_k = \begin{bmatrix} B_{k0} \\ B_{k1} \\ \vdots \\ B_{kn} \end{bmatrix}, \quad \alpha = \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_n \end{bmatrix} \)

(25)

\[
X_k = \begin{bmatrix}
W_k(\omega)\left[\Omega_0(\omega), \Omega_2(\omega), \ldots, \Omega_n(\omega)\right] \\
\vdots \\
W_k(\omega_{N_f})\left[\Omega_0(\omega_{N_f}), \Omega_2(\omega_{N_f}), \ldots, \Omega_n(\omega_{N_f})\right]
\end{bmatrix}
\]

(26)

\[
Y_k = \begin{bmatrix}
-W_k(\omega)\left[\Omega_0(\omega)H_2(\omega), \Omega_2(\omega)H_2(\omega), \ldots, \Omega_n(\omega)H_2(\omega)\right] \\
\vdots \\
-W_k(\omega_{N_f})\left[\Omega_0(\omega_{N_f})H_2(\omega_{N_f}), \Omega_2(\omega_{N_f})H_2(\omega_{N_f}), \ldots, \Omega_n(\omega_{N_f})H_2(\omega_{N_f})\right]
\end{bmatrix}
\]

(27)

Note that every equation in (23) has been weighted with a frequency-dependent function \( W_j(\omega_j) \). The quality of the estimates can often be improved by using an adequate weighting function. The (complex) Jacobian matrix \( J \) of this least-squares problem

\[
J = \begin{bmatrix}
X_1 & 0 & \cdots & 0 & Y_1 \\
0 & X_2 & 0 & \cdots & Y_2 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & X_{N_o N_i} & Y_{N_o N_i}
\end{bmatrix}
\]

(28)

has \( N_f N_o N_i \) rows and \( (n+1)(N_o N_i +1) \) columns (with \( N_f >> n \), where \( n \) is the order of the polynomials, \( n = 2N_m \) ).
3.1.3 Normal Equations

Many estimators used in modal analysis form the normal equations explicitly, i.e. they compute \( \text{Re}(J^H J) \) explicitly. Note that the real part of \( J^H J \) has to be taken because the coefficients are real. Deriving the estimates directly from the Jacobian matrix leads to a better-conditioned problem. However, forming the normal equations can result in a faster implementation, as will be the case here too. The normal equations can be written as

\[
\begin{bmatrix}
R_1 & 0 & \cdots & S_1 \\
0 & R_2 & & S_2 \\
\vdots & & \ddots & \vdots \\
S_1^T & S_2^T & \cdots & \sum_{k=1}^{N_f} T_k
\end{bmatrix}
\begin{bmatrix}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_{N_f N_i}
\end{bmatrix}
\approx 0
\]

(29)

with \( R_k = \text{Re}(X_k^H X_k) \), \( S_k = \text{Re}(X_k^H Y_k) \), and \( T_k = \text{Re}(Y_k^H Y_k) \). The entries of these matrices equal

\[
R_k(r,s) = \text{Re}\left( \sum_{f=1}^{N_f} W_k(\omega_f) \Omega_s^H(\omega_f) \Omega_{r,s}(\omega_f) \right)
\]

\[
S_k(r,s) = -\text{Re}\left( \sum_{f=1}^{N_f} W_k(\omega_f) H_k(\omega_f) \Omega_s^H(\omega_f) \Omega_{r,s}(\omega_f) \right)
\]

\[
T_k(r,s) = \text{Re}\left( \sum_{f=1}^{N_f} W_k(\omega_f) H_k(\omega_f) \Omega_s^H(\omega_f) \Omega_{r,s}(\omega_f) \right)
\]

(30)

If a discrete time-domain model is used, i.e. \( \Omega_j(\omega_f) = \exp(-i \omega_f T_s \cdot j) \), and if the frequencies are uniformly distributed (i.e. \( \omega_f = f \cdot \Delta \omega, \ f = 1, \ldots, N_f \), with \( \Delta \omega = 2\pi/NT_s \)), then, the above summations can be rewritten as

\[
R_k(r,s) = \text{Re}\left( \sum_{f=1}^{N_f} W_k(\omega_f) \Omega_s^H(\omega_f) \Omega_{r,s}(\omega_f) \right)
\]

\[
S_k(r,s) = -\text{Re}\left( \sum_{f=1}^{N_f} W_k(\omega_f) H_k(\omega_f) \Omega_s^H(\omega_f) \Omega_{r,s}(\omega_f) \right)
\]

\[
T_k(r,s) = \text{Re}\left( \sum_{f=1}^{N_f} W_k(\omega_f) H_k(\omega_f) \Omega_s^H(\omega_f) \Omega_{r,s}(\omega_f) \right)
\]

(31)

One can readily verify that the above matrices have a Toeplitz structure and that their entries can be time-efficiently computed with the Fast Fourier Transform (FFT) algorithm.

3.1.4 Reduced Normal Equations

Although the number of rows of the normal matrix in (29) is much smaller than the number of rows of the Jacobian matrix (28), its size is still quite huge (i.e. \( (n+1)(N_o N_i + 1) \) rows and columns). As we are mainly interested in a fast and stable method to construct a stabilization chart (see next section), only the denominator coefficients (i.e. the poles) are in fact required. Elimination of the numerator coefficients

\[
\beta_k = - R_k^{-1} \cdot S_k \cdot \alpha
\]

(32)

yields

\[
\left[ \sum_{k=1}^{N_f N_i} T_k - S_k^T \cdot R_k^{-1} \cdot S_k \right] \cdot \alpha \approx 0
\]

(33)
or \( \mathbf{M} \cdot \alpha = 0 \) with \( \mathbf{M} = \sum_{k=1}^{N_i N_i} \mathbf{T}_k - \mathbf{S}_{k}^T \cdot \mathbf{R}_{k}^{-1} \cdot \mathbf{S}_{k} \). The size of the square matrix \( \mathbf{M} \) is \( n+1 \), and thus much smaller than the original normal equation (29). To remove the parameter redundancy of transfer function model (20) (and to avoid the trivial solution with all coefficient equal to zero), a constraint has to be imposed on the coefficients of the transfer functions. This can be done, for instance, by imposing that one of the coefficients is equal to a non-zero constant value. Assume, for instance, that the last coefficient of \( \alpha \) is constrained to 1 (i.e. coefficient \( n+1 \)). In that case, the Least Squares (LS) estimate of \( \alpha \) is given by

\[
\hat{\alpha}_{LS} = \left\{ -[\mathbf{M}(1:n,1:n)]^{-1} \cdot \{\mathbf{M}(1:n,n+1)\} \right\}_1
\]

Once \( \hat{\alpha}_{LS} \) is known, (32) can be used to derive all \( \hat{\beta}_{LS} \) coefficients. This approach is more time efficient than solving (29) directly (approximately \( N_o^2 N_i^2 \) times faster). The mode shape vectors are derived using (14) and (15).

### 3.1.5 Stabilization chart

In modal analysis, a stabilization chart is an important tool that is often used to assist the user in separating physical poles from mathematical ones. A stabilization chart is obtained by repeating the analysis for increasing model order \( n \). For each model order, the poles are calculated from the estimated denominator coefficients. The stable poles (i.e. the poles with a negative real part) are then presented graphically in ascending model order in a so-called “stabilization chart” (see Figure 6). Estimated poles corresponding to physically relevant system modes tend to appear for each estimation order at nearly identical locations, while the so-called mathematical poles, i.e. poles resulting from the mathematical solution of the normal equations but meaningless with respect to the physical interpretation, tend to jump around. These mathematical poles are mainly due to the presence of noise on the measurements.

![Figure 6. Stabilization chart obtained with (a) a time-domain estimator (LSCE) and (b) the frequency-domain least-squares estimator.](image)

The LSCE estimator (Least Squares Complex Exponential) is probably the most frequently used technique in industry. The LSCE estimator is a time-domain technique that makes use of impulse response functions (16) to derive the modal parameters. In Figure 6 the stabilization chart of the LSCE estimator is compared with the proposed frequency-domain least squares estimator. It turns out that in many applications, the frequency-domain estimator is able to generate quite clear stabilization compared to the LSCE approach.
### 3.2 Maximum Likelihood Estimation

#### 3.2.1 Gauss-Newton Optimization

Assuming the FRFs to be uncorrelated, the (negative) log-likelihood function reduces to

\[
\ell_{\text{ML}}(\theta) = \sum_{k=1}^{N} \sum_{f=1}^{F} \frac{\left| \hat{H}_k(\theta, \omega_f) - H_k(\omega_f) \right|^2}{\text{var}(H_k(\omega_f))} \tag{35}
\]

The Maximum Likelihood (ML) estimate of \( \theta = [\beta_1^T, \ldots, \beta_{N,N}^T, \alpha^T]^T \) is obtained by minimizing (35). This can be done by means of a Gauss-Newton optimization algorithm, which takes advantage of the quadratic form of the cost function (35). The Gauss-Newton iterations are given by

\[
\begin{align*}
(\text{a}) & \quad \text{solve } \text{Re}(J_p^H J_p) \delta_p = -\text{Re}(J_p^H r_p) \text{ for } \delta_p \\
(\text{b}) & \quad \text{set } \theta_{p+1} = \theta_p + \delta_p
\end{align*}
\tag{36}
\]

with \( r_p = r(\theta_p) \), \( J_p = \partial r(\theta)/\partial \theta|_{\theta_p} \), and

\[
r(\theta) = \begin{bmatrix}
\frac{\hat{H}_{11}(\theta, \omega_1) - H_{11}(\omega_1)}{\sqrt{\text{var}(H_{11}(\omega_1))}} \\
\vdots \\
\frac{\hat{H}_{N,N}(\theta, \omega_N) - H_{N,N}(\omega_N)}{\sqrt{\text{var}(H_{N,N}(\omega_N))}}
\end{bmatrix} \tag{37}
\]

The Jacobian matrix \( J_p \) has the same structure as the matrix \( J \) given in (28). Also here it is possible to form the normal equations (i.e. \( \text{Re}(J_p^H J_p) \) and \( \text{Re}(J_p^H r_p) \)) in a similar time-efficient way as presented in Section 3.1. See 6.43.8.2 Estimation with Known Noise Model and 6.43.8.4 Estimation with Unknown Noise Model for more information about frequency-domain ML identification.

#### 3.2.2 Confidence Intervals

A good approximation of the covariance matrix of the ML estimate \( \hat{\theta}_{\text{ML}} \) is obtained by inverting the Fisher information matrix (see 6.43.8. Frequency Domain System Identification)

\[
\text{cov}\{\hat{\theta}_{\text{ML}}\} \approx [2 \text{Re}(J_{\omega}^H J_{\omega})]^{-1} \tag{38}
\]

with \( J_{\omega} \) the Jacobian matrix evaluated in the last iteration step of the Gauss-Newton optimization.

As one is mainly interested in the uncertainty on the modal frequencies and damping ratios, only the covariance matrix of the denominator coefficients is in fact required. Starting from (38), one can show that this matrix is given by

\[
\text{cov}\{\hat{\alpha}_{\text{ML}}\} \approx \left[ 2 \sum_{k=1}^{N} T_k - S_k^T \cdot R_k^{-1} \cdot S_k \right]^{-1} \tag{39}
\]

with \( R_k \), \( S_k \), and \( T_k \) as defined in Section 3.1.3 but now applied to \( \text{Re}(J_{\omega}^H J_{\omega}) \). Hence, it is not necessary to invert the full matrix occurring in (38). From (39), it is possible to compute the uncertainty on the modal frequencies and damping ratios. For flight flutter testing – but also for applications such as vibration-based fault detection and operational modal analysis – the availability of reliable estimates together with confidence intervals is important.
4. Application

In this section, modal analysis will be applied to a slat track, which is safety critical component of an airplane. Slat tracks (see Figure 7) are located at the leading edge of an aircraft wing and make part of a gliding mechanism that is used to enlarge the wing surface. The enlargement of the wing surface is needed in order to increase the lift force at reduced velocity during landing and take off. A slat track of an Airbus A320 airplane is considered here. The A320 airplane has 5 slats per wing. The first slat (i.e. the inboard slat between fuselage and engine) contains 4 slat tracks. The other 4 slats have 2 slat tracks each. Safety critical components such as slat tracks are rigorously tested to prove their ability to withstand all safety regulations. It is commonly accepted that the track should outlive the plane by five times. Using computer simulations, it is possible to predict the lifetime of a track using Finite Element (FE) models. To validate the dynamic behavior of these FE models, experimentally obtained estimates of the modal parameters are required.

![Figure 7: Mounted slat track on an Airbus A320 in extended position.](image)

In Figure 8(a) a shaker is used to excite the slat track with a multisine signal. A multisine is a periodic signal consisting of a sum of sine waves at uniformly distributed frequencies (from 0 Hz to 8192 Hz in this case with a resolution of 1 Hz). A force sensor is attached on the slat track and is in contact with the shaker through a stinger. The stinger has the characteristic of being stiff in only one direction, i.e. that of the intended excitation. In this example, one surface of the track is
measured by means of a scanning laser Doppler vibrometer (see Figure 8(b); the shaker is at the backside of the slat track, which is vertically suspended by means of a elastic rope). The FRF measurements are performed in 500 points uniformly distributed over the surface of one side of the slat track (see Figure 8(c)).

Starting from the FRF measurements, the modal parameters can be estimated using the frequency-domain estimator given in Section 3. In the considered frequency band, a few hundreds of modes are present. Figure 9 shows 4 of the estimated mode shapes (with enlarged displacement amplitudes). The real displacements are of the order of a few micrometers.

![Figure 9: Mode shapes of the slat track for a modal frequency of (a) 4091 Hz, (b) 4161 Hz, (c) 4739 Hz, and (d) 5169 Hz](image-url)
5. Conclusions

In this contribution about modal analysis, a ‘dedicated’ multivariable implementation for frequency-domain estimators, based on a common-denominator transfer function model, has been given. Typical for modal analysis is the very large number of outputs. This huge amount of data requires dedicated algorithms that balance between accuracy and memory/computing requirements. The results given in this contribution can be generalized to other frequency-domain estimators such as the Total Least Squares (TLS), Generalized Total Least Squares (GTLS), Iterative Quadratic Maximum Likelihood (IQML), … (see 6.43.8.2 Estimation with Known Noise Model and 6.43.8.4 Estimation with Unknown Noise Model).

Bibliography

Ewins D.J. (2000). Modal Testing: Theory, Practice and Application, Hertfordshire: Research Studies Press. [In this book, all the steps involved in planning, executing, interpreting and applying the results from a modal test are described in straightforward terms.]


