Proceedings of IDETC/CIE 2005
ASME 2005 International Design Engineering Technical Conferences &
Computers and Information in Engineering Conference
September 24-28, 2005, Long Beach, USA

DETC2005-84917

GENERAL FORMULATION OF AN EFFICIENT RECURSIVE ALGORITHM BASED ON CANONICAL MOMENTA FOR FORWARD DYNAMICS OF CLOSED-LOOP MULTIBODY SYSTEMS

Joris Naudet
Vrije Universiteit Brussel
Faculty of Applied Sciences
Department of Mechanical Engineering
Pleinlaan 2, 1050 Brussel, Belgium
Email: joris.naudet@vub.ac.be

Dirk Lefeber
Vrije Universiteit Brussel
Faculty of Applied Sciences
Department of Mechanical Engineering
Pleinlaan 2, 1050 Brussel, Belgium
Email: dlefeber@vub.ac.be

ABSTRACT

In previous work, a method for establishing the equations of motion of open-loop multibody mechanisms was introduced. The proposed forward dynamics formulation resulted in a Hamiltonian set of $2n$ first order ODE's in the generalized coordinates q and the canonical momenta p. These Hamiltonian equations were derived from a recursive Newton-Euler formulation. It was shown how an $O(n)$ formulation could be obtained in the case of a serial structure with general joints. The amount of required arithmetical operations was considerably less than comparable acceleration based formulations. In this paper, a further step is taken: the method is extended to constrained multibody systems. Using the principle of virtual power, it is possible to obtain a recursive Hamiltonian formulation for closed-loop mechanisms as well, enabling the combination of the low amount of arithmetical operations and a better evolution of the constraints violation errors, when compared with acceleration based methods.

INTRODUCTION

One could state that multibody dynamics research is focused on two major challenges nowadays. The first challenge is the increase of simulation speed by calculating the equations of motion in a more efficient way or by creating better numerical integrators. The second challenge is about the efficient incorporation of events in the simulation: contacts, impacts, changing topology, user interaction... These events require a high flexibility of the simulator and can use a great amount of computer processing time, certainly in the case of contact detection for a high number of bodies or bodies with a complex structure. This paper is a contribution to the first challenge.

There exist many ways to treat the equations of motion, but the recursive formulations have proven to be very efficient for large numbers of bodies [1]. Most methods are acceleration based: whether the Newton-Euler equations, the Lagrangian equations or the principle of virtual work or virtual power are used, second order differential equations are obtained and the algorithms come down to calculating and integrating accelerations [1–8].

An important aspect is the set of coordinates that describes the state of the system, because it does have strong repercussions on the numerical integration. Expressing the equations of motion in a minimal set of coordinates results in less differential equations, which are however more coupled and usually exhibit stronger non-linearities [9], compared to the non-minimal formulations. The advantage is that no constraint equations are required and a set of ordinary differential equations (ODE) must be solved. Non-minimal formulations on the other hand, result in mixed differential algebraic equations(DAE), but are in general much easier to establish.

Instead of manipulating the number of coordinates, one can also change the nature of the coordinates. Acceleration based

formulations require starting values for the generalized coordinates and the velocities. In that sense, one can also think of these equations as first order differential equations in the generalized coordinates and the velocities. An interesting alternative are Hamilton's equations, which are expressed in terms of the generalized coordinates and their *conjugated canonical momenta*. Promising, because these equations behave better during numerical integration, resulting in more accuracy and stability [10, 11]. Despite that fact, the Hamiltonian formulation is not often encountered in multibody dynamics literature. The reason for the lack of interest is probably that the construction of Hamilton's equations is computationally intensive and cannot compete with the recursive acceleration based algorithms, even with the advantageous behavior during the numerical integration. A few researchers [12, 13] devoted time to the use of the Hamiltonian equations in multibody systems dynamics and obtained very positive results.

In previous work [14], an additional step to promote the use of canonical momenta was taken, by introducing a new *recursive* method to establish Hamilton's equations for open-loop (rigid) multibody systems. The presented algorithm did not only provide an Hamiltonian $O(n)$ equivalent for the acceleration based methods, but even exceeded their performance at the level of number of required arithmetical operations.

The problem of obtaining the equations of motion becomes more involved when additional constraints are applied on the system, as is the case with closed-loop systems [15, 16]. The strong interdependency of the coordinates and their velocities makes it difficult to tailor a $O(n)$ recursive algorithm, certainly to obtain Hamiltonian equations. It is however possible, and that will be shown in this article.

The paper is further divided in two parts. In the first part, the basic formalism of the method is introduced and the algorithm for open-loop systems is briefly reviewed. The second part tackles the problem of additional constraints.

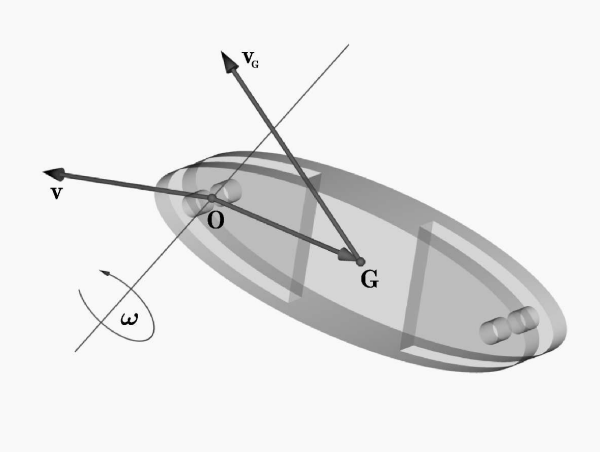

Figure 1. KINEMATICS NOTATION ON A RIGID BODY.

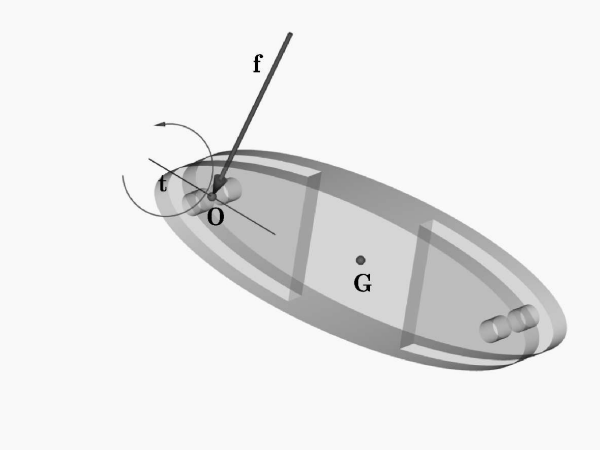

Figure 2. DYNAMICS NOTATION ON A RIGID BODY.

NEWTON-EULER IN RELATIVE AXES

The classical formulation of the Newton-Euler equations for a single rigid body is given by

$$m\frac{d^0\mathbf{v}_G}{dt} = \mathbf{f} + \mathbf{f}_r \tag{1a}$$

$$\mathbf{J}_G\frac{d^K\omega}{dt} + \omega \times \mathbf{J}_G\omega = \mathbf{t}_G + \mathbf{t}_{r_G} \tag{1b}$$

The first equation is typically written in an inertial reference frame (notation $\frac{d^0}{dt}$), while the second is formulated in a frame K fixed to the body ($\frac{d^K}{dt}$). The force and the torque that act on the object are represented by $\mathbf{f}$ and $\mathbf{t}$, the *reaction* forces and torques by $\mathbf{f}_r$ and $\mathbf{t}_r$. The matrix $\mathbf{J}$ is the inertia tensor, m is the mass of the body, ω is the angular velocity referred to the inertial axes and $\mathbf{v}_G$ the linear velocity of the center of mass (see figure 1). The index G denotes that the momenta and the tensor of inertia are taken with respect to the center of mass.

The 6-dimensional momentum *vector* will be needed, it is defined as follows:

$$\mathbf{P} = \begin{pmatrix}\mathbf{p}_l \\ \mathbf{p}_a\end{pmatrix} = \begin{pmatrix} m\mathbf{I} & m\widetilde{\mathbf{GO}} \\ m\widetilde{\mathbf{OG}} & \mathbf{J}\end{pmatrix}\begin{pmatrix}\mathbf{v} \\ \omega\end{pmatrix}_K = \mathbf{M\Omega} \tag{2}$$

Inspection of $\mathbf{P}$ reveals that it is nothing more than a concatenation of the linear ($\mathbf{p}_l$) and angular ($\mathbf{p}_a$) momenta of the rigid body. $\mathbf{I}$ is a unity dyadic, $\mathbf{v}$ the linear velocity of the origin O of

the local reference frame. This origin must lie on the rotational joint axes, if present. $\mathbf{J}$ is the tensor of inertia referred to point O. $\mathbf{M}$ is called the mass matrix. $\tilde{\mathbf{x}}$ is a skew-symmetric matrix constructed from the vector $\mathbf{x}$ and is an alternative notation for the cross product.

$$\mathbf{x} \times \mathbf{a} = \tilde{\mathbf{x}}\,\mathbf{a} = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \tag{3}$$

Ω is the spatial velocity vector. It can be written as a function of the coordinate velocities (scleronomic constraints):

$$\Omega = \mathbf{E}\dot{\mathbf{q}} \tag{4}$$

We call $\mathbf{E}$ the joint matrix. The column vectors of the joint matrix form a basis for the space of virtual motions and are hence orthogonal to the space of the generalized reaction forces. They are the partial derivatives of the spatial velocity vector to the generalized coordinates. The coordinate velocities vector $\dot{\mathbf{q}}$ has dimension n, which is the number of degrees of freedom of the body. The joint matrix therefore has dimensions $6 \times n$. In case of rheonomic constraints, an additional term Ω_t is required to account for the prescribed motion.
The Newton-Euler equations (1) can be reformulated in relative axes, and written with respect to the origin O. Note that the relation between the time derivatives in two different frames K and L is given by

$$\frac{d^L\mathbf{x}}{dt} = \frac{d^K\mathbf{x}}{dt} + \omega_r \times \mathbf{x} \tag{5}$$

ω_r being the relative angular velocity of frame K with respect to frame L.
Furthermore, the momentum vector (2) can be introduced in the equations. After some mathematical manipulations, and observing that $\mathbf{p}_l = m\mathbf{v}_G$, equations (1) can be reformulated as:

$$\begin{pmatrix} \dot{\mathbf{p}}_l \\ \dot{\mathbf{p}}_a \end{pmatrix} + \begin{pmatrix} \tilde{\omega} & \mathbf{0} \\ \tilde{\mathbf{v}} & \tilde{\omega} \end{pmatrix} \begin{pmatrix} \mathbf{p}_l \\ \mathbf{p}_a \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{t} \end{pmatrix} + \begin{pmatrix} \mathbf{f}_r \\ \mathbf{t}_r \end{pmatrix} \tag{6}$$

By convention, all momenta are taken with respect to the origin O of the local reference frame. $\dot{\mathbf{x}}$ stands for the time derivative in *local* axes, e.g. $\dot{\omega}_K = \frac{d^K\omega_K}{dt}$. This implies that $\dot{\mathbf{M}}_K = 0$.

We will go further in the conciseness of the equations, by defining a 6-dimensional cross product as follows:

$$\Omega\times = \begin{pmatrix} \mathbf{v} \\ \omega \end{pmatrix} \times \triangleq \begin{pmatrix} \tilde{\omega} & \mathbf{0} \\ \tilde{\mathbf{v}} & \tilde{\omega} \end{pmatrix} \tag{7}$$

The equations of motion for a single rigid body then become

$$\dot{\mathbf{P}} + \Omega \times \mathbf{P} = \mathbf{T} + \mathbf{T}_r \tag{8}$$

with $\mathbf{T} = \begin{pmatrix} \mathbf{f}^T & \mathbf{t}^T \end{pmatrix}^T$, $\mathbf{T}_r = \begin{pmatrix} \mathbf{f}_r^T & \mathbf{t}_r^T \end{pmatrix}^T$.

HAMILTONIAN EQUATIONS

Introducing the Hamiltonian equations requires a brief description of the Lagrange equations. These are given by [17]:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} + \Phi_q^T\lambda = \mathbf{Q} \tag{9a}$$

$$\Phi(\mathbf{q}, t) = \mathbf{0} \tag{9b}$$

This is a set of differential algebraic equations (DAE). The differential equations are of order 2. Φ are the constraints equations. The Lagrange equations are described by the set $(\mathbf{q}, \dot{\mathbf{q}})$, which are the coordinates and their velocities. Using the so-called Legendre transformation, it is possible to transform this set of coordinates into the same $\mathbf{q}$ and their conjugated *canonical momenta* $\mathbf{p}$, which are defined as:

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} \tag{10}$$

They are an extension of the concept of linear and angular momenta to generalized coordinates. Applying the Legendre transformation results in

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}} \tag{11a}$$

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}} + \mathbf{Q} - \Phi_{\mathbf{q}}^T\lambda \tag{11b}$$

$$\Phi(\mathbf{q}, t) = 0 \tag{11c}$$

Referring to the alternative formulation of the Newton-Euler equations, it can easily be shown that the kinetic energy T of a single rigid body can be expressed as:

$$T = \frac{1}{2}\Omega^T \mathbf{M}\Omega = \frac{1}{2}\Omega^T \mathbf{P} \tag{12}$$

Calculating the canonical momenta with (10) yields

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} = \frac{\partial T}{\partial \dot{\mathbf{q}}} = \frac{\partial \Omega^T}{\partial \dot{\mathbf{q}}}\mathbf{M}\Omega = \mathbf{E}^T\mathbf{M}\Omega = \mathbf{E}^T\mathbf{P} \tag{13}$$

The canonical momenta $\mathbf{p}$ conjugated to the generalized coordinates $\mathbf{q}$ are thus the projections of the momentum vector $\mathbf{P}$ on the joint axes.

EQUATIONS OF MOTION FOR OPEN-LOOP MULTIBODY SYSTEMS

In this section, a short overview of the algorithm for open-loop multibody systems will be given. For a detailed derivation and description, take a look at [14, 18, 19].

Force And Velocity Transformations

By convention, the reactions (torques) from body N are taken with respect to point O_N on the joint axis. To transmit these reactions to origin O_K of body K, the transformation matrix ${}^K\mathcal{T}_N^F$ is used:

$${}^K\mathcal{T}_N^F = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \widetilde{\mathbf{O_K O_N}} & \mathbf{I} \end{pmatrix} \tag{14}$$

Note that this matrix is constant in the local reference frame. Observe also that the velocities transform in a similar way:

$$\Omega_N = {}^N\mathcal{T}_K^V \Omega_K + \mathbf{E}_N\dot{\mathbf{q}}_N = \begin{pmatrix} \mathbf{I} & \widetilde{\mathbf{O_N O_K}} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{v}_K \\ \omega_K \end{pmatrix} + \mathbf{E}_N\dot{\mathbf{q}}_N \tag{15}$$

The relationship between both transformation matrices is given by:

$${}^K\mathcal{T}_N^F = ({}^N\mathcal{T}_K^V)^T \tag{16}$$

Articulated Momentum Vector

The articulated momentum vector $\mathbf{P}^*$ of a rigid body K in a multibody system is defined as the sum of the momentum vector of that body and the reduced momentum vectors of *all* its *outboard* bodies. This is equivalent to freezing all outboard links of the considered body and calculating the momentum vector of the obtained *articulated* structure. The articulated momentum vectors can be derived with a backward recursion:

$$\mathbf{P}_K^* = \mathbf{P}_K + \sum_i {}^K\mathcal{T}_i^F \mathbf{P}_i \quad i \in \{\text{outboard bodies}\} \tag{17}$$

$$\mathbf{P}_K^* = \mathbf{P}_K + \sum_j {}^K\mathcal{T}_j^F \mathbf{P}_j^* \quad j \in \{\text{adjacent outboard bodies}\} \tag{18}$$

It can also be expressed as:

$$\mathbf{P}_K^* = \mathbf{M}_K^*\Omega_K + \mathbf{D}_K \tag{19}$$

with the so called articulated mass matrix $\mathbf{M}^*$ and the remainder momentum vector $\mathbf{D}$. These quantities can be obtained in a backward recursion step.

$$\mathbf{M}_K^* = \mathbf{M}_K + \sum_j {}^K\mathcal{T}_j^F \mathbf{M}_j'^{j}\mathcal{T}_K^V \tag{20}$$

$$\mathbf{D}_K = \sum_j {}^K\mathcal{T}_j^F \mathbf{D}_j' \qquad j \in \{\text{adjacent outboard bodies}\} \tag{21}$$

$$\mathbf{M}_K' = \mathbf{M}_K^* - \mathbf{M}_K^*\mathbf{E}_K\mathbf{M}_{j_K}^{-1}\mathbf{E}_K^T\mathbf{M}_K^* \tag{22}$$

$$\mathbf{M}_{j_K} = \mathbf{E}_K^T\mathbf{M}_K^*\mathbf{E}_K \tag{23}$$

$$\mathbf{D}_K' = \mathbf{M}_K^*\mathbf{E}_K\mathbf{M}_{j_K}^{-1}(\mathbf{p}_K - \mathbf{E}_K^T\mathbf{D}_K) + \mathbf{D}_K \tag{24}$$

Canonical Momenta

It can straightforwardly be proved that the projection of the articulated momentum vector on the subspace of virtual motion of a certain joint results in a set of canonical momenta conjugated to the coordinates describing that motion.

$$\mathbf{p}_K = \frac{\partial L}{\partial \dot{\mathbf{q}}_K} = \sum_{i=K}^{N} \frac{\partial \Omega_i^T}{\dot{\mathbf{q}}_K}\mathbf{P}_i = \mathbf{E}_K^T \sum_{i=K}^{N} {}^K\mathcal{T}_i^F \mathbf{P}_i = \mathbf{E}_K^T\mathbf{P}_K^* \tag{25}$$

Equations Of Motion

Using the equations of motion (8) for a single rigid body and the concept of articulated momentum vector, the equations of motion for each body of a MBS can be obtained:

$$\dot{\mathbf{P}}_K^* + \Omega_K \times \mathbf{P}_K^* = \mathbf{T}_K^* + \mathbf{T}_{r_K} \tag{26}$$

The unknown reaction forces can be eliminated by projection on the subspace $\mathbf{E}_K$:

$$\dot{\mathbf{p}}_K = \mathbf{E}_K^T(\mathbf{T}_K^* - \Omega_K \times \mathbf{P}_K^*) + \dot{\mathbf{E}}_K^T \mathbf{P}_K^* \tag{27}$$

The coordinate velocities can be found using (19) in a forward recursion step:

$$\dot{\mathbf{q}}_K = \mathbf{M}_{j_K}^{-1} \mathbf{E}_K^T [(\mathbf{P}_K - \mathbf{D}_K) - \mathbf{M}_K^{*K} \mathcal{T}_{K-1}^V \Omega_{K-1}] \tag{28}$$

EQUATIONS OF MOTION FOR CONSTRAINED MBS

The principle of the method for obtaining the recursive Hamiltonian equations for a constrained MBS will be shown through an example, as a general description would be too long and tedious. The method is based on reductions of dependent spatial velocities [15, 16] to independent spatial velocities. The relative (or joint) coordinates are partitioned in a set of independent and a set of dependent coordinates, which results in a minimal formulation. Note that the canonical momenta are only defined for the independent coordinates. The choice of dependent coordinates is arbitrary, but should always be valid: singularities can occur and should be avoided. In this example however, the dependent coordinates are chosen at the tip of the underlying open-loop structure. This simplifies the calculations. There is a loss in generality, but the purpose of the paper is to present the basic ideas of the algorithm, not to explain it in full details. One may revert to [20] for a general discussion of acceleration based equations.

The considered example is a chain of bodies interconnected by pin-joints, the base body 1 being connected to a fixed inertial frame 0. A closed loop is created by connecting the last element N of the chain to the fixed inertial frame C. The number of links N is arbitrary, but must be more than two.

Principle Of Virtual Power

The principle of virtual power states that reaction forces acting on a mechanical system do not deliver any power under a *virtual* motion [21]. It can be expressed under following form:

$$\sum_i [\Omega_i^{*T}(\dot{\mathbf{P}}_i + \Omega_i \times \mathbf{P}_i - \mathbf{T}_i)] = 0, \tag{29}$$

Ω^* being the *virtual* spatial velocities. These equations can be written as functions of the *virtual* coordinate velocities $\dot{\mathbf{q}}^*$ by using (15):

$$\sum_{i=1}^{N} \mathbf{A}_i \dot{\mathbf{q}}_i^* = \mathbf{0} \tag{30}$$

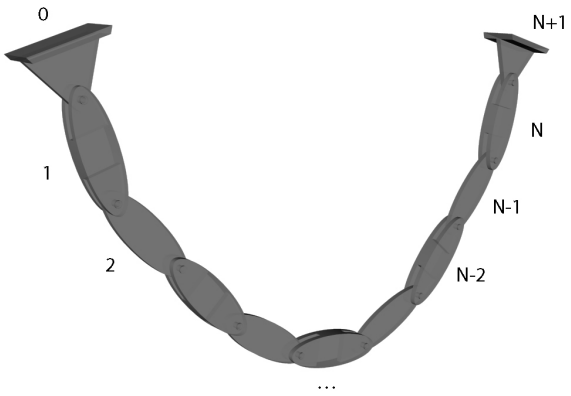

Figure 3. EXAMPLE OF A CONSTRAINED MBS.

These equations must be fulfilled for every set of allowed virtual coordinate velocities. For unconstrained systems, this means the coefficients $\mathbf{A}$ can all be set to zero, leading to N first order differential equations $\mathbf{A}_i = \mathbf{0}$. In the case of constrained systems with DOF degrees of freedom, a partition can be made in dependent and independent coordinates. Expressing the dependent virtual coordinate velocities as functions of the independent ones gives:

$$\sum_i \mathbf{B}_i \dot{\mathbf{q}}_i^* = \mathbf{0} \qquad i \in \{\text{independent bodies}\} \tag{31}$$

The coefficients $\mathbf{B}$ can now be set to zero and DOF first order differential equations are obtained.

Jacobian Of The Constraint Equations

The relation between dependent and independent coordinates is given implicitly by the constraint equations:

$$\Phi(\mathbf{q}) = \mathbf{0} \tag{32}$$

Dealing with these equations directly is not an easy task, this is why their time-derivatives are often taken, leading to a relationship between the coordinate *velocities*.

$$\Phi_{\mathbf{q}} \dot{\mathbf{q}} = -\frac{\partial \Phi}{\partial t} \tag{33}$$

$\Phi_{\mathbf{q}}$ is the Jacobian matrix of the constraint equations. After partitioning in dependent $\mathbf{q}_d$ and independent $\mathbf{q}_i$ coordinates, one

gets:

$$\Phi_{\mathbf{q}_d}\dot{\mathbf{q}}_d + \Phi_{\mathbf{q}_i}\dot{\mathbf{q}}_i = -\frac{\partial \Phi}{\partial t} \tag{34}$$

$$\dot{\mathbf{q}}_\mathbf{d} = -\Phi_{\mathbf{q}_d}^{-1}(\Phi_{\mathbf{q}_i}\dot{\mathbf{q}}_i + \frac{\partial \Phi}{\partial t}), \tag{35}$$

which gives the relationship between the dependent and the independent coordinate velocities. Needless to say that an incorrect partitioning will lead to singularity or at least bad conditioning of $\Phi_{\mathbf{q}_d}$.

Using the Jacobian matrix to obtain equations (31) results in expressions for the coefficients $\mathbf{B}$ which are of order n^2 and which introduce a high coupling of the equations of motion. It is therefore unsuitable for the goal to achieve a recursive $O(n)$ method.

Dependent Spatial Velocities

Instead of using the Jacobian in an explicit way, one can describe the spatial velocity of a *dependent body* (which joint coordinates are chosen as dependent ones) as a function of the spatial velocity of the adjacent inboard body [16]. As will be shown, this leads naturally to an expression for the canonical momenta and to an $O(n)$ method to obtain the Hamiltonian equations of motion.

The example on fig.3 has $N-2$ degrees of freedom. Closing the loop by connecting N to C introduces 2 constraints and one extra joint, which will be described by joint coordinate $\mathbf{q}_C$. As a consequence, there are 3 dependent coordinates which will be chosen as $\mathbf{q}_{N-1}$, $\mathbf{q}_N$ and $\mathbf{q}_C$. Body C is fixed, its spatial velocity is therefore zero:

$$\Omega_C = {}^C\mathcal{T}_N^V\Omega_N + \mathbf{E}_C\dot{\mathbf{q}}_C = \mathbf{0} \tag{36}$$

After projection on subspace $\mathbf{E}_C$, an expression for joint velocity C is obtained:

$$\dot{\mathbf{q}}_C = -(\mathbf{E}_C^T\mathbf{E}_C)^{-1}\mathbf{E}_C^{T\,C}\mathcal{T}_N^V\Omega_N \tag{37}$$

$$= \mathbf{C}_{\mathbf{q}_C}^{T\ C}\mathcal{T}_N^V\Omega_N \tag{38}$$

Substitution in (36) results in

$$\mathbf{C}_C{}^C\mathcal{T}_N^V\Omega_N = \mathbf{0} \tag{39}$$

with

$$\mathbf{C}_C = \mathbf{I} + \mathbf{E}_C\mathbf{C}_{\mathbf{q}_C}^T \tag{40}$$

This procedure can be repeated recursively for all dependent coordinates. For the next body N, one gets (premultiplying by $\mathbf{E}_N^{T\,N}\mathcal{T}_C^F$ this time)

$$\dot{\mathbf{q}}_N = -(\mathbf{E}_N^{T\,N}\mathcal{T}_C^F\mathbf{C}_C{}^C\mathcal{T}_N^V\mathbf{E}_N)^{-1}(\mathbf{E}_N^{T\,N}\mathcal{T}_C^F\mathbf{C}_C{}^C\mathcal{T}_N^V)^N\mathcal{T}_{N-1}^V\Omega_{N-1}$$
$$= \mathbf{C}_{\mathbf{q}_N}^{T\ N}\mathcal{T}_{N-1}^V\Omega_{N-1} \tag{41}$$

Note that matrix $(\mathbf{E}_N^{T\,N}\mathcal{T}_C^F\mathbf{C}_C{}^C\mathcal{T}_N^V\mathbf{E}_N)$ needs to be regular. Singularity would be the consequence of a bad partitioning in independent and dependent coordinates. After substitution in (15), one gets

$$\Omega_N = \mathbf{C}_N{}^N\mathcal{T}_{N-1}^V\Omega_{N-1} \tag{42}$$

with

$$\mathbf{C}_N = \mathbf{I} + \mathbf{E}_N\mathbf{C}_{\mathbf{q}_N}^T \tag{43}$$

There is one more dependent coordinate to find. Substitution of (42) in (39) and premultiplying by ${}^N\mathcal{T}_C^F$ results in

$$\mathbf{C}_N^{*\,N}\mathcal{T}_{N-1}^V\Omega_{N-1} = \mathbf{0} \tag{44}$$

with

$$\mathbf{C}_N^* = {}^N\mathcal{T}_C^F\mathbf{C}_C{}^C\mathcal{T}_N^V\mathbf{C}_N \tag{45}$$

$$= \Lambda_N\mathbf{C}_N \tag{46}$$

which is a symmetrical matrix. Further calculations yield:

$$\dot{\mathbf{q}}_{N-1} = \mathbf{C}_{\mathbf{q}_{N-1}}^{T\ N-1}\mathcal{T}_{N-2}^V\Omega_{N-2} \tag{47}$$

$$\Omega_{N-1} = \mathbf{C}_{N-1}{}^{N-1}\mathcal{T}_{N-2}^V\Omega_{N-2} \tag{48}$$

with

$$\mathbf{C}_{\mathbf{q}_{N-1}}^T = -(\mathbf{E}_{N-1}^{T\ N-1}\mathcal{T}_N^F\mathbf{C}_N^{*\,N}\mathcal{T}_{N-1}^V\mathbf{E}_{N-1})^{-1}(\mathbf{E}_{N-1}^{T\ N-1}\mathcal{T}_N^F\mathbf{C}_N^{*\,N}\mathcal{T}_{N-1}^V)$$
$$\mathbf{C}_{N-1} = \mathbf{I} + \mathbf{E}_{N-1}\mathbf{C}_{\mathbf{q}_{N-1}}^T \tag{49}$$

The constraints matrices $\mathbf{C}^*$ and $\mathbf{C}_\mathbf{q}^T$ are found through a backward recursion step, the joint and spatial velocities through a forward recursion step. Note that $\mathbf{CC} = \mathbf{C}$, this means it is a projection operator. Note also that $\mathbf{CE} = \mathbf{0}$.

Canonical Momenta

The canonical momenta of a constrained multibody system are only defined for the independent coordinates. There are in this case thus $N-2$ canonical momenta, which can be found with:

$$\mathbf{p}_K = \frac{\partial T}{\partial \dot{\mathbf{q}}_K} = \sum_{i=1}^{N} \frac{\partial \Omega_i^T}{\partial \dot{\mathbf{q}}_K} \mathbf{P}_i = \sum_{i=ind} \frac{\partial \Omega_i^T}{\partial \dot{\mathbf{q}}_K} \mathbf{P}_i + \sum_{i=dep} \frac{\partial \Omega_i^T}{\partial \dot{\mathbf{q}}_K} \mathbf{P}_i \quad (50)$$
$$K \in \{\text{independent bodies}\}$$

Using (15), (42) and (48) yields

$$\mathbf{p}_K = \mathbf{E}_K^T \mathbf{P}_K^c \quad (51)$$

with the so called constrained momentum vector

$$\mathbf{P}_K^c = \mathbf{P}_K + \sum_j {}^K\mathcal{T}_j^F \mathbf{C}_j^T \mathbf{P}_j^c \qquad j \in \{\text{adjacent outboard bodies}\} \quad (52)$$

$\mathbf{C}_j^T$ is set to unity for independent bodies.

Equations Of Motion

The equations of motion of the constrained MBS will be found using the principle of virtual power. To obtain a suitable expression (31), one needs to write the virtual spatial velocities explicitly as functions of the independent virtual joint velocities. When going from the tip C to the base 0, the first encountered independent coordinate is $\mathbf{q}_{N-2}$. Following spatial velocities are dependent on $\dot{\mathbf{q}}_{N-2}$:

$$\Omega_{N-2} = {}^{N-2}\mathcal{T}_{N-3}^V \Omega_{N-3} + \mathbf{E}_{N-2}\dot{\mathbf{q}}_{N-2} \quad (53)$$
$$\Omega_{N-1} = \mathbf{C}_{N-1}{}^{N-1}\mathcal{T}_{N-2}^V \Omega_{N-2} \quad (54)$$
$$\Omega_N = \mathbf{C}_N{}^N\mathcal{T}_{N-1}^V \Omega_{N-1} \quad (55)$$

Substitution in the principle of virtual power leads to following expression for coefficient $\mathbf{B}_{N-2}$:

$$\begin{aligned}\mathbf{B}_{N-2} &= \mathbf{E}_{N-2}^T(\dot{\mathbf{P}}_{N-2} + \Omega_{N-2}\times\mathbf{P}_{N-2} - \mathbf{T}_{N-2}) \quad (56)\\ &+ \mathbf{E}_{N-2}^T{}^{N-2}\mathcal{T}_{N-1}^F \mathbf{C}_{N-1}^T(\dot{\mathbf{P}}_{N-1} + \Omega_{N-1}\times\mathbf{P}_{N-1} - \mathbf{T}_{N-1})\\ &+ \mathbf{E}_{N-2}^T{}^{N-2}\mathcal{T}_{N-1}^F \mathbf{C}_{N-1}^T{}^{N-1}\mathcal{T}_N^F \mathbf{C}_N^T(\dot{\mathbf{P}}_N + \Omega_N\times\mathbf{P}_N - \mathbf{T}_N) = \mathbf{0}\end{aligned}$$

After the introduction of the constrained momentum vector (52), some tedious manipulations and a lot of perseverance, it can be proved that following equality holds:

$$\begin{aligned}&\dot{\mathbf{P}}_{N-1} + \Omega_{N-1}\times\mathbf{P}_{N-1} - \mathbf{T}_{N-1} + {}^{N-1}\mathcal{T}_N^F \mathbf{C}_N^T(\dot{\mathbf{P}}_N + \Omega_N\times\mathbf{P}_N - \mathbf{T}_N)\\ &= \dot{\mathbf{P}}_{N-1}^c + \Omega_{N-1}\times\mathbf{P}_{N-1}^c - \mathbf{T}_{N-1}^c \quad (57)\end{aligned}$$

with

$$\begin{aligned}\mathbf{T}_{N-1}^c &= \mathbf{T}_{N-1} + {}^{N-1}\mathcal{T}_N^F \mathbf{C}_N^T \mathbf{T}_N\\ &+ {}^{N-1}\mathcal{T}_N^F[\dot{\mathbf{C}}_N^T + (\Omega_N\times\mathbf{I})\mathbf{C}_N^T - \mathbf{C}_N^T(\Omega_N\times\mathbf{I})]\mathbf{P}_N \quad (58)\end{aligned}$$

A comparable reduction can be made from body $N-1$ to body $N-2$, ultimately resulting in the concise and familiar form

$$\dot{\mathbf{p}}_{N-2} = \mathbf{E}_{N-2}^T(\mathbf{T}_{N-2}^c - \Omega_{N-2}\times\mathbf{P}_{N-2}^c) + \dot{\mathbf{E}}_{N-2}^T\mathbf{P}_{N-2}^c \quad (59)$$

with

$$\begin{aligned}\mathbf{T}_{N-2}^c &= \mathbf{T}_{N-2} + {}^{N-2}\mathcal{T}_{N-1}^F \mathbf{C}_{N-1}^T \mathbf{T}_{N-1}^c \quad (60)\\ &+ {}^{N-2}\mathcal{T}_{N-1}^F[\dot{\mathbf{C}}_{N-1}^T + (\Omega_{N-1}\times\mathbf{I})\mathbf{C}_{N-1}^T\mathbf{C}_{N-1}^T(\Omega_{N-1}\times\mathbf{I})]\mathbf{P}_{N-1}^c\end{aligned}$$

All the other bodies K can be handled as for in open-loop systems:

$$\dot{\mathbf{p}}_K = \mathbf{E}_K^T(\mathbf{T}_K^c - \Omega_K\times\mathbf{P}_K^c) + \dot{\mathbf{E}}_K^T\mathbf{P}_K^c \quad (61)$$
$$\mathbf{T}_K^c = \mathbf{T}_K + {}^K\mathcal{T}_{K+1}^F \mathbf{T}_{K+1}^c \quad (62)$$

$\mathbf{T}_i^c$ should of course be calculated in a second backward recursion, as it is dependent on the spatial velocities, which are calculated in the forward recursion step. This extra recursion step is often needed when the forces are velocity dependent anyway.

Coordinate Velocities

The independent coordinate velocities are needed to obtain the remaining Hamiltonian equations. The dependent coordinate velocities $\dot{\mathbf{q}}_{N-1}$, $\dot{\mathbf{q}}_N$ and $\dot{\mathbf{q}}_C$ were already calculated in the section about dependent spatial velocities. To find joint velocity $\dot{\mathbf{q}}_{N-2}$, the projection of the constrained momentum vector on the joint axis is needed.

$$\mathbf{p}_{N-2} = \mathbf{E}_{N-2}^T \mathbf{P}_{N-2}^c = \mathbf{E}_{N-2}^T (\mathbf{P}_{N-2} + {}^{N-2}\mathcal{T}_{N-1}^F \mathbf{C}_{N-1}^T \mathbf{P}_{N-1}^c) \quad (63)$$

Each term needs to be expressed as a function of Ω_{N-2}. For body $N-1$, one obtains:

$$\begin{aligned}
\mathbf{P}_{N-1}^c &= \mathbf{P}_{N-1} + {}^{N-1}\mathcal{T}_{N-1}^F \mathbf{C}_N^T \mathbf{P}_N \\
&= (\mathbf{M}_{N-1} + {}^{N-1}\mathcal{T}_N^F \mathbf{C}_N^T \mathbf{M}_N \mathbf{C}_N {}^N\mathcal{T}_{N-1}^V)\Omega_{N-1} \\
&= \mathbf{M}_{N-1}^c \Omega_{N-1} = \mathbf{M}_{N-1}^c \mathbf{C}_{N-1} {}^{N-1}\mathcal{T}_{N-2}^V \Omega_{N-2} \quad (64) \\
\mathbf{M}_{N-1}^c &= \mathbf{M}_{N-1} + {}^{N-1}\mathcal{T}_N^F \mathbf{C}_N^T \mathbf{M}_N \mathbf{C}_N {}^N\mathcal{T}_{N-1}^V \quad (65)
\end{aligned}$$

$\mathbf{M}_{N-1}^c$ being the constrained mass matrix. For body $N-2$, one subsequently gets:

$$\begin{aligned}
\mathbf{P}_{N-2}^c &= [\mathbf{M}_{N-2} + {}^{N-2}\mathcal{T}_{N-1}^F \mathbf{C}_{N-1}^T \mathbf{M}_{N-1}^c \mathbf{C}_{N-1} {}^{N-1}\mathcal{T}_{N-2}^V]\Omega_{N-2} \\
&= \mathbf{M}_{N-2}^c \Omega_{N-2} \quad (66) \\
\mathbf{M}_{N-2}^c &= \mathbf{M}_{N-2} + {}^{N-2}\mathcal{T}_{N-1}^F \mathbf{C}_{N-1}^T \mathbf{M}_{N-1}^c \mathbf{C}_{N-1} {}^{N-1}\mathcal{T}_{N-2}^V \quad (67)
\end{aligned}$$

The joint velocity $\dot{\mathbf{q}}_{N-2}$ can easily be derived from above equations:

$$\begin{aligned}
\mathbf{p}_{N-2} &= \mathbf{E}_{N-2}^T \mathbf{P}_{N-2}^c \\
&= \mathbf{E}_{N-2}^T \mathbf{M}_{N-2}^c ({}^{N-2}\mathcal{T}_{N-3}^V \Omega_{N-3} + \mathbf{E}_{N-2}\dot{\mathbf{q}}_{N-2}) \quad (68) \\
\dot{\mathbf{q}}_{N-2} &= (\mathbf{E}_{N-2}^T \mathbf{M}_{N-2}^c \mathbf{E}_{N-2})^{-1} (\mathbf{p}_{N-2} - \mathbf{E}_{N-2}^T \mathbf{M}_{N-2}^c {}^{N-2}\mathcal{T}_{N-3}^V \Omega_{N-3}) \\
&= \mathbf{M}_{j_{N-2}}^{c^{-1}} [\mathbf{p}_{N-2} - \mathbf{E}_{N-2}^T \mathbf{M}_{N-2}^c {}^{N-2}\mathcal{T}_{N-3}^V \Omega_{N-3}] \quad (69)
\end{aligned}$$

To obtain the joint velocity for body $N-3$, one can substitute above equation in the expression for the constrained momentum vector.

$$\begin{aligned}
\mathbf{P}_{N-2}^c &= \mathbf{M}_{N-2}^c ({}^{N-2}\mathcal{T}_{N-3}^V \Omega_{N-3} + \mathbf{E}_{N-2}\dot{\mathbf{q}}_{N-2}) \\
&= \mathbf{M}_{N-2}^c [{}^{N-2}\mathcal{T}_{N-3}^V \Omega_{N-3} \\
&\quad + \mathbf{E}_{N-2}\mathbf{M}_{j_{N-2}}^{c^{-1}} (\mathbf{p}_{N-2} - \mathbf{E}_{N-2}^T \mathbf{M}_{N-2}^c {}^{N-2}\mathcal{T}_{N-3}^V \Omega_{N-3})] \\
&= \mathbf{M}_{N-2}^{'} {}^{N-2}\mathcal{T}_{N-3}^V \Omega_{N-3} + \mathbf{D}_{N-2}^{'} \quad (70) \\
\mathbf{M}_{N-2}^{'} &= \mathbf{M}_{N-2}^c - \mathbf{M}_{N-2}^c \mathbf{E}_{N-2} \mathbf{M}_{j_{N-2}}^{c^{-1}} \mathbf{E}_{N-2}^T \mathbf{M}_{N-2}^c \quad (71) \\
\mathbf{D}_{N-2}^{'} &= \mathbf{M}_{N-2}^c \mathbf{E}_{N-2} \mathbf{M}_{j_{N-2}}^{c^{-1}} \mathbf{p}_{N-2} \quad (72)
\end{aligned}$$

Repeating previous procedure gives:

$$\begin{aligned}
\mathbf{P}_{N-3}^c &= \mathbf{P}_{N-3} + {}^{N-3}\mathcal{T}_{N-2}^F \mathbf{P}_{N-2}^c \\
&= (\mathbf{M}_{N-3} + {}^{N-3}\mathcal{T}_{N-2}^F \mathbf{M}_{N-2}^{'} {}^{N-2}\mathcal{T}_{N-3}^V)\Omega_{N-3} + {}^{N-3}\mathcal{T}_{N-2}^F \mathbf{D}_{N-2}^{'} \\
&= \mathbf{M}_{N-3}^c \Omega_{N-3} + \mathbf{D}_{N-3} \quad (73) \\
\mathbf{M}_{N-3}^c &= \mathbf{M}_{N-3} + {}^{N-3}\mathcal{T}_{N-2}^F \mathbf{M}_{N-2}^{'} {}^{N-2}\mathcal{T}_{N-3}^V \quad (74) \\
\mathbf{D}_{N-3} &= {}^{N-3}\mathcal{T}_{N-2}^F \mathbf{D}_{N-2}^{'} \quad (75)
\end{aligned}$$

The joint velocity is then:

$$\begin{aligned}
\mathbf{p}_{N-3} &= \mathbf{E}_{N-3}^T \mathbf{P}_{N-3}^c \quad (76) \\
&= \mathbf{E}_{N-3}^T \mathbf{M}_{N-3}^c ({}^{N-3}\mathcal{T}_{N-4}^V \Omega_{N-4} + \mathbf{E}_{N-3}\dot{\mathbf{q}}_{N-3}) + \mathbf{E}_{N-3}^T \mathbf{D}_{N-3} \\
\dot{\mathbf{q}}_{N-3} &= (\mathbf{E}_{N-3}^T \mathbf{M}_{N-3}^c \mathbf{E}_{N-3})^{-1} (\mathbf{p}_{N-3} - \mathbf{E}_{N-3}^T \mathbf{D}_{N-3} \\
&\quad - \mathbf{E}_{N-3}^T \mathbf{M}_{N-3}^c {}^{N-3}\mathcal{T}_{N-4}^V \Omega_{N-4}) \\
&= \mathbf{M}_{j_{N-3}}^{c^{-1}} (\mathbf{p}_{N-3} - \mathbf{d}_{N-3} - \mathbf{E}_{N-3}^T \mathbf{M}_{N-3}^c {}^{N-3}\mathcal{T}_{N-4}^V \Omega_{N-4}) \quad (77)
\end{aligned}$$

All the other joint velocities can be found just like for open-loop systems.

SUMMARY OF THE ALGORITHM

Before calculating the Hamiltonian equations of motion, a partition must be made in dependent and independent coordinates. This should be done carefully, considering singularity conditions. The actual algorithm is divided in 3 recursion steps. In a first, backward recursion step, the constraint matrices $\mathbf{C}$, the constrained mass matrices $\mathbf{M}^c$ and the remainder momentum vectors $\mathbf{D}$ are computed. In the following, forward recursion step, all coordinate velocities $\dot{\mathbf{q}}$ and all spatial velocity vectors Ω are calculated. In a last, backward recursion step, the accumulated force vectors $\mathbf{T}^c$ are obtained, from which the time derivatives of the canonical momenta $\dot{\mathbf{p}}$ can be found.

CONCLUSIONS

In this paper, it was shown through a simple example how an $O(n)$ recursive Hamiltonian algorithm can be obtained for a constrained multibody system. The use of the Hamiltonian equations of motion has a positive influence on the evolution of the constraint violation errors, as constraints are introduced at velocity level instead of acceleration level. Additionally, the algorithm is based on its open-loop variant, which proved to be more efficient than recursive acceleration based algorithms when comparing the number of required arithmetical operations to obtain the equations of motion.

ACKNOWLEDGMENT

This research is supported by the Institute for the Promotion of Innovation by Science and Technology in Flanders (Belgium).

REFERENCES

[1] Featherstone, R., 1987. *Robot Dynamics Algorithms*. Kluwer Academic Publishers.

[2] Jerkovsky, W., 1978. "The Structure of Multibody Dynamics Equations". *Journal of Guidance and Control*, **1**(3), pp. 173–182.

[3] Kane, T., and Levinson, D., 1985. *Dynamics: Theory and Applications*. McGraw-Hill.

[4] Haug, E., 1989. *Computer-Aided Kinematics and Dynamics of Mechanical Systems. Volume I: Basic Methods*. Allyn and Bacon.

[5] Rosenthal, D., 1990. "An Order *n* Formulation for Robotic Systems". *The Journal of the Astronautical Sciences*, **38**(4), October-December, pp. 511–529.

[6] Rein, U., 1993. "Efficient Object Oriented Programming of Multibody Dynamics Formalisms". *Proceedings of the NATO-Advanced Study Institute on Computer Aided Analysis of Rigid and Flexible Mechanical Systems*, **2**, pp. 59–69.

[7] Vukobratović, M., Filaretov, V., and Korzun, A., 1994. "A unified approach to mathematical modelling of robotic manipulator dynamics". *Robotica*, **12**, pp. 411–420.

[8] Baraff, D., 1996. "Linear-Time Dynamics using Lagrange multipliers". *Computer Graphics Proceedings, Annual Conference Series*, pp. 137–146.

[9] de Jalón, J. G., and Bayo, E., 1994. *Kinematic and Dynamic Simulation of Multibody Systems: The Real-Time Challenge*. Springer-Verlag, New York.

[10] Brenan, K., Campbell, S., and Petzold, L., 1989. *The Numerical Solution of Initial Value Problems in Differential-Algebraic Equations*. Elsevier Science Publishing.

[11] Baumgarte, J., 1983. "A New Method of Stabilization for Holonomic Constraints". *ASME Journal of Applied Mechanics*, **50**, pp. 869–870.

[12] Lankarani, H., and Nikravesh, P., 1988. "Application of the Canonical Equations of Motion in Problems of Constrained Multibody Systems with Intermittent Motion". *Advances in Design Automation*, **1**, pp. 417–423.

[13] Bayo, E., and Avello, A., 1994. "Singularity-Free Augmented Lagrangian Algorithms for Constrained Multibody Dynamics". *Nonlinear Dynamics*, **5**, pp. 209–231.

[14] Naudet, J., Lefeber, D., and Terze, Z., 2003. "General Formulation of an Efficient Recursive Algorithm based on Canonical Momenta for Forward Dynamics of Open-Loop Multibody Systems". *ECCOMAS Thematic Conference on Multibody Systems*, pp. 2003–33.

[15] Stejskal, V., and Valášek, M., 1996. *Kinematics and Dynamics of Machinery*. Marcel Dekker.

[16] Anderson, K., and Critchley, J., 2003. "Order-N Performance Algorithm for the Simulation of Constrained Multi-Rigid-Body Dynamic Systems". *Multibody System Dynamics*, **9**, pp. 185–212.

[17] Pars, L., 1965. *A Treatise of Analytical Dynamics*. William Heinemann.

[18] Lefeber, D., Naudet, J., Terze, Z., and Daerden, F., 2002. "Forward dynamics of multibody mechanisms using an efficient algorithm based on canonical momenta". *NATO ASI Workshop on Virtual Nonlinear Multibody Systems*, **1**, pp. 121–126.

[19] Naudet, J., Lefeber, D., and Terze, Z., 2003. "Forward Dynamics of Open-Loop Multibody Mechanisms Using an Efficient Recursive Algorithm Based on Canonical Momenta". *Multibody Systems Dynamics*, **10**(1), pp. 45–59.

[20] Critchley, J., and Anderson, K., 2003. "A generalized recursive coordinate reduction method for multibody dynamic systems". *Journal of Multiscale Computational Engineering*, **1**(2), pp. 181–200.

[21] Goldstein, H., 1950. *Classical Mechanics*. Addison-Wesley Publishing Company.