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GENERAL FORMULATION OF AN EFFICIENT RECURSIVE ALGORITHM BASED ON CANONICAL MOMENTA FOR FORWARD DYNAMICS OF OPEN-LOOP MULTIBODY SYSTEMS

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Abstract: *A new method for establishing the equations of motion of multibody mechanisms based on canonical momenta is introduced in this paper. In absence of constraints, the proposed forward dynamics formulation results in a Hamiltonian set of $2n$ first order ODE's in the generalized coordinates q and the canonical momenta p. These Hamiltonian equations are derived from a recursive Newton-Euler formulation. It is shown how in the case of a serial structure with general joints, an $O(n)$ formulation is obtained. The amount of required arithmetical operations is considerably less than comparable acceleration based formulations.*

1 INTRODUCTION

During the last decades, a lot of research has been done to find new algorithms, new numerical integration techniques and better implementation methods to speed up the dynamical simulation of complex multibody mechanisms. Amongst many others, Featherstone,[10] Kane and Levinson,[9] Rosenthal,[12] Vukobratović,[14] Jerkovsky[8] and Baraff[15] put significant efforts in finding efficient order N methods to derive the equations of motion. Lankarani,[6] Bayo and Avello[7] developed techniques to integrate these equations in a stable and efficient way. Work has also been done to implement algorithms on a

parallel computing architecture (Bae et al.[16]). All this research and the fast evolution of computer technology resulted in quite fast simulations nowadays. These simulations, however, involve mechanisms of ever increasing complexity (large amount of parts, flexibility, friction, backlash) and demand an ever increasing accuracy and, hence, number of computations. It is therefore interesting to continue this research in order to find more efficient algorithms. In previous publications[17,],[18] we took a step in that direction and presented a new, canonical momenta based algorithm, which allowed a speedup of simulations by reducing the number of operations required to obtain the equations of motion. That algorithm was however tailored for mechanical systems with rotational joints with one degree of freedom. In this paper, a more general approach is adopted to allow any kind of kinematic constraints between the bodies, rheonomic ones included.

2 LAGRANGE'S EQUATIONS

The equations of motion for a multibody system described by n generalized coordinates q and having m holonomic constraint conditions can be found using the well-known Lagrangian approach[1,2,]:[5]

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\mathbf{q}}}) - \frac{\partial L}{\partial \mathbf{q}} + \mathbf{\Phi}_q^T \boldsymbol{\lambda} = \mathbf{Q} \tag{1a}$$

$$\mathbf{\Phi}(\mathbf{q}, t) = \mathbf{0} \tag{1b}$$

$L = T - V$ is the Lagrangian function, where T and V are respectively the total kinetic and the total potential energies of the system. $\mathbf{Q}$ are the external generalized forces, $\mathbf{\Phi}$ are the constraint equations, $\mathbf{\Phi}_q$ is the Jacobian matrix of these constraints and $\boldsymbol{\lambda}$ are the Lagrange multipliers. Equations (1) form a mixed set of differential algebraic equations (DAE). To solve this set using conventional numerical integration schemes, it has to be turned onto a set of ordinary differential equations (ODE). This can be done by differentiating the constraint equations (1b) twice with respect to time:

$$\mathbf{\Phi_q}\ddot{\mathbf{q}} + \dot{\mathbf{\Phi}}_\mathbf{q}\dot{\mathbf{q}} + \dot{\mathbf{\Phi}}_t = \mathbf{0} \tag{2}$$

$\mathbf{\Phi}_t$ is the partial derivative of the constraints with respect to time. In multibody system dynamics, the resulting ODE is typically put in following matrix notation:

$$\begin{pmatrix} \mathbf{M} & \mathbf{\Phi}_q^T \\ \mathbf{\Phi}_q & \mathbf{0} \end{pmatrix} \begin{pmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{Q} + L_\mathbf{q} - \dot{\mathbf{M}}\dot{\mathbf{q}} \\ -\dot{\mathbf{\Phi}}_\mathbf{q}\dot{\mathbf{q}} - \dot{\mathbf{\Phi}}_t \end{pmatrix} \tag{3}$$

$\mathbf{M}$ is the mass matrix. The use of the constraint equations at the acceleration level (2) does not have any theoretical repercussion, it does however induce problems during numerical integration. As small numerical errors are introduced on the acceleration level,

these will be integrated twice and will result in uncontrolled errors on the velocity and position levels. Therefore the need for stabilization methods for the constraint violation errors. Well known procedures are the Baumgarte[3] stabilization method and the coordinate partitioning method.[11] Promising alternatives using a manifold approach where proposed by Blajer[20] and Terze.[19]

3 HAMILTON'S EQUATIONS

The Hamiltonian equations can be found by applying a Legendre transformation on the Lagrangian.[1] This transformation changes the description of the system in terms of generalized coordinates $\mathbf{q}$ and velocities $\dot{\mathbf{q}}$ to a description in terms of the same coordinates $\mathbf{q}$ and their conjugated canonical momenta $\mathbf{p}$. These canonical momenta are defined as:

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} \tag{4}$$

They are an extension of the concept of linear and angular momenta to generalized coordinates. Applying the Legendre transformation yields

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}} \tag{5a}$$

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}} + \mathbf{Q} - \Phi_{\mathbf{q}}^T \boldsymbol{\lambda} \tag{5b}$$

$$\boldsymbol{\Phi}(\mathbf{q}, t) = 0 \tag{5c}$$

This is a set of DAE's with $2n$ first order differential equations and m kinematic constraint equations. $H = \mathbf{p}^T \dot{\mathbf{q}} - L$ is the Hamiltonian function. DAE's are characterized by a so-called differential index. The acceleration based formulations have an index of 3, the Hamiltonian formulation has index 2.[5] As shown by Brenan et al.,[4] index 2 DAE's have a better behavior during numerical integration. Hence, the use of canonical momenta may be numerically advantageous compared to the use of accelerations. Transforming DAE (5) into a set of ODE's, in analogy with the Lagrangian formulation, leads to the same conclusion. This can be seen by considering the equations of motion as the solution of a variational problem with constraints[6,7]. This implicates the definition of the so-called augmented Lagrangian which includes the constraints on the velocity level:

$$L^* = L + \dot{\boldsymbol{\Phi}}^T \boldsymbol{\sigma} \tag{6}$$

It results in following set of equations:

$$\begin{pmatrix} \mathbf{M} & \boldsymbol{\Phi}_q^T \\ \boldsymbol{\Phi}_q & \mathbf{0} \end{pmatrix} \begin{pmatrix} \dot{\mathbf{q}} \\ \boldsymbol{\sigma} \end{pmatrix} = \begin{pmatrix} \mathbf{p} \\ -\boldsymbol{\Phi}_t \end{pmatrix} \tag{7}$$

together with

$$\dot{\mathbf{p}} = L_{\mathbf{q}} + \mathbf{Q} + \dot{\mathbf{\Phi}}_{\mathbf{q}}^{T}\boldsymbol{\sigma} \tag{8}$$

As the constraints were used at the velocity level, the numerical errors are integrated only once, resulting in smaller constraint violations.
This paper however describes an effort to construct sets of Hamiltonian equations *for serial structures*. When using the joint coordinates as generalized coordinates, no constraint equations (5c) are needed and the last term of (5b) disappears. The forward dynamic problem can be stated as to evaluate functions $\mathbf{G}_1$ ans $\mathbf{G}_2$ in following general form:

$$\dot{\mathbf{p}} = G_1(\mathbf{q}, \mathbf{p}, t) \tag{9a}$$
$$\dot{\mathbf{q}} = G_2(\mathbf{q}, \mathbf{p}, t) \tag{9b}$$

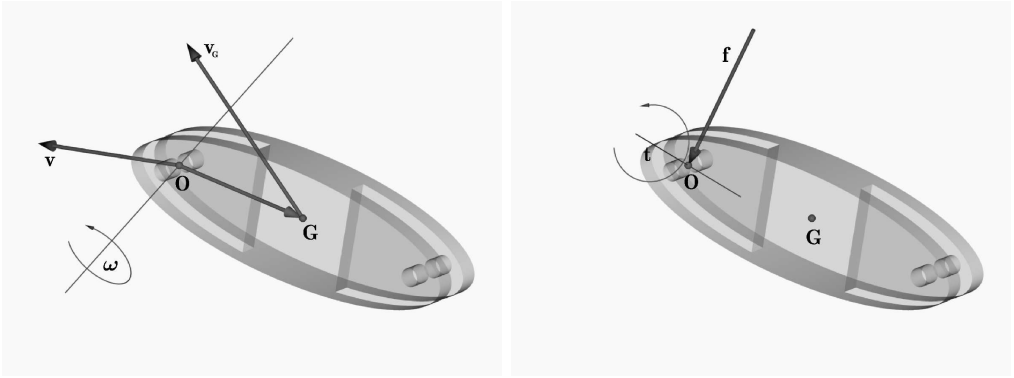
Hamiltonian equations are computationally intensive to derive straightforwardly, for the Hamilton function H has to be established from the Lagrangian function L which already requires a considerable amount of arithmetical operations. This is probably the reason for the lack of interest in Hamilton's equations in the domain of multibody mechanics. In most acceleration based $O(n)$ algorithms, the equations of motion are found by recursion. This way the direct derivation of the Lagrangian *function* L and its partial derivatives are avoided and much faster evaluations of the equations of motion are obtained. However, it also seems possible to find an $O(n)$ algorithm based on canonical momenta. That algorithm, as will be shown in the following sections, has a reduced number of operations, compared even to the most efficient acceleration based algorithms. This advantage and the improved numerical behavior makes it a very promising alternative.

4 NEWTON-EULER IN RELATIVE AXES

The classical formulation of the Newton-Euler equations for a single free moving body is given by

$$m\frac{d^0\mathbf{v}_G}{dt} = \mathbf{f} \tag{10a}$$
$$\mathbf{J}_G\frac{d^K\boldsymbol{\omega}}{dt} + \boldsymbol{\omega} \times \mathbf{J}_G\boldsymbol{\omega} = \mathbf{t}_G \tag{10b}$$

The first equation is typically written in an inertial reference frame (notation $\frac{d^0}{dt}$), while the second is formulated in a frame K fixed to the body ($\frac{d^K}{dt}$). The force and the torque that act on the object are represented by $\mathbf{f}$ and $\mathbf{t}$. The matrix $\mathbf{J}$ is the inertia tensor, m is the mass of the body, $\boldsymbol{\omega}$ is the angular velocity referred to the inertial axes and $\mathbf{v}_G$ the linear velocity of the center of mass (see figure 1). The index G denotes that the momenta and the tensor of inertia are taken with respect to the center of mass.



(a) Kinematics (b) Dynamics

Figure 1: Notation on a single rigid body

The 6-dimensional momentum *vector* will be needed, it is defined as follows:

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_l \\ \mathbf{p}_a \end{pmatrix} = \begin{pmatrix} m\mathbf{I} & m\widetilde{\mathbf{GO}} \\ m\widetilde{\mathbf{OG}} & \mathbf{J} \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\omega} \end{pmatrix}_K = \mathbf{M}\boldsymbol{\Omega} \tag{11}$$

This is *not* the same vector as was used in the previous section to denote the canonical momenta $\mathbf{p}$. Inspection of $\mathbf{P}$ reveals that it is nothing more than a concatenation of the linear ($\mathbf{p}_l$) and angular ($\mathbf{p}_a$) momenta of the rigid body. $\mathbf{I}$ is a unity matrix, $\mathbf{v}$ the linear velocity of the origin O of the local reference frame. This origin must lie on the rotational joint axes, if present. $\mathbf{J}$ is the tensor of inertia referred to point O. $\mathbf{M}$ is here also called mass matrix, but it is not the same as in previous section. In the remainder of the article, only this mass matrix will be referred to. $\tilde{\mathbf{x}}$ is a skew-symmetric matrix constructed from the vector $\mathbf{x}$ and is an alternative notation for the cross product.

$$\mathbf{x} \times \mathbf{a} = \tilde{\mathbf{x}}\,\mathbf{a} = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \tag{12}$$

$\boldsymbol{\Omega}$ is the spatial velocity vector. It can be written as a function of the coordinate velocities:

$$\boldsymbol{\Omega} = \mathbf{E}_j \dot{\mathbf{q}} + \boldsymbol{\Omega}_t \tag{13}$$

We call $\mathbf{E}_j$ the joint matrix. The column vectors of the joint matrix form a basis for the space of virtual motions and are hence orthogonal to the space of the generalized reaction forces. They are the partial derivatives of the spatial velocity vector to the generalized coordinates. The coordinate velocities vector $\dot{\mathbf{q}}$ has dimension n, which is the number of degrees of freedom of the body. The joint matrix therefore has dimensions $6 \times n$. $\boldsymbol{\Omega}_t$ is

the partial derivative of the spatial velocity vector with respect to time. It disappears when scleronomic constraints are considered.
Instead of trying to find an algorithm directly starting from the Hamiltonian equations, the Newton-Euler equations (10) are reformulated in relative axes, and written with respect to the origin O. Note that the relation between the time derivatives in two different frames K and L is given by

$$\frac{d^L\mathbf{x}}{dt} = \frac{d^K\mathbf{x}}{dt} + \boldsymbol{\omega}_r \times \mathbf{x} \tag{14}$$

$\boldsymbol{\omega}_r$ is the relative angular velocity of frame K with respect to frame L.
Furthermore, the momentum vector (11) is introduced in the equations. After some mathematical manipulations, and observing that $\mathbf{p}_l = m\mathbf{v}_G$, equations (10) can be reformulated as:

$$\begin{pmatrix}\dot{\mathbf{p}}_l \\ \dot{\mathbf{p}}_a\end{pmatrix} + \begin{pmatrix}\tilde{\boldsymbol{\omega}} & \mathbf{0} \\ \tilde{\mathbf{v}} & \tilde{\boldsymbol{\omega}}\end{pmatrix}\begin{pmatrix}\mathbf{p}_l \\ \mathbf{p}_a\end{pmatrix} = \begin{pmatrix}\mathbf{f} \\ \mathbf{t}\end{pmatrix} \tag{15}$$

By convention, all momenta are taken with respect to the origin O of the local reference frame. $\dot{\mathbf{x}}$ stands for the time derivative in *local* axes, e.g. $\dot{\boldsymbol{\omega}}_K = \frac{d^K\boldsymbol{\omega}_K}{dt}$. This means that $\dot{\mathbf{M}}_K = 0$.

We will go further in the conciseness of the equations, by defining a 6-dimensional cross product as follows:

$$\boldsymbol{\Omega}\times = \begin{pmatrix}\mathbf{v} \\ \boldsymbol{\omega}\end{pmatrix}\times \triangleq \begin{pmatrix}\tilde{\boldsymbol{\omega}} & \mathbf{0} \\ \tilde{\mathbf{v}} & \tilde{\boldsymbol{\omega}}\end{pmatrix} \tag{16}$$

The equations of motion for a single rigid body then become

$$\dot{\mathbf{P}} + \boldsymbol{\Omega} \times \mathbf{P} = \mathbf{T} \tag{17}$$

with $\mathbf{T} = \begin{pmatrix}\mathbf{f} & \mathbf{m}\end{pmatrix}^T$.
It can easily be shown that the kinetic energy T of a single rigid body can be expressed as:

$$T = \frac{1}{2}\boldsymbol{\Omega}^T\mathbf{M}\boldsymbol{\Omega} = \frac{1}{2}\boldsymbol{\Omega}^T\mathbf{P} \tag{18}$$

Calculating the canonical momenta with (4) yields

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} = \frac{\partial T}{\partial \dot{\mathbf{q}}} = \frac{\partial \boldsymbol{\Omega}^T}{\partial \dot{\mathbf{q}}}\mathbf{M}\boldsymbol{\Omega} = \mathbf{E}_j^T\mathbf{M}\boldsymbol{\Omega} = \mathbf{E}_j^T\mathbf{P} \tag{19}$$

The canonical momenta $\mathbf{p}$ conjugated to the generalized coordinates $\mathbf{q}$ are thus the projections of the momentum vector $\mathbf{P}$ on the joint axes.

5 FIRST SET OF EQUATIONS (G_1)

5.1 Articulated momentum vector

In this section we will derive one set of equations depicted in (9), namely the one involving the evaluation of the function G_1. The other set will be discussed in the next section. Rewriting (17) for the last body N and splitting the external generalized forces in the known external part $\mathbf{T}_N$ and the unknown reaction part $\mathbf{T}_{r_N}$ —resulting from the interaction with the previous body $N-1$— gives

$$\dot{\mathbf{P}}_N + \boldsymbol{\Omega}_N \times \mathbf{P}_N = \mathbf{T}_N + \mathbf{T}_{r_N} \tag{20}$$

For the remaining inboard bodies, the equations become more involved, as there are two locations where reactions occur (fig. 2). Body $K = N-1$ is connected with bodies $K-1$ and N. Therefore the equations of motion can be written as

$$\dot{\mathbf{P}}_K + \boldsymbol{\Omega}_K \times \mathbf{P}_K = \mathbf{T}_K + \mathbf{T}_{r_K} - {}^K\mathcal{T}_N^F \mathbf{T}_{r_N} \tag{21}$$

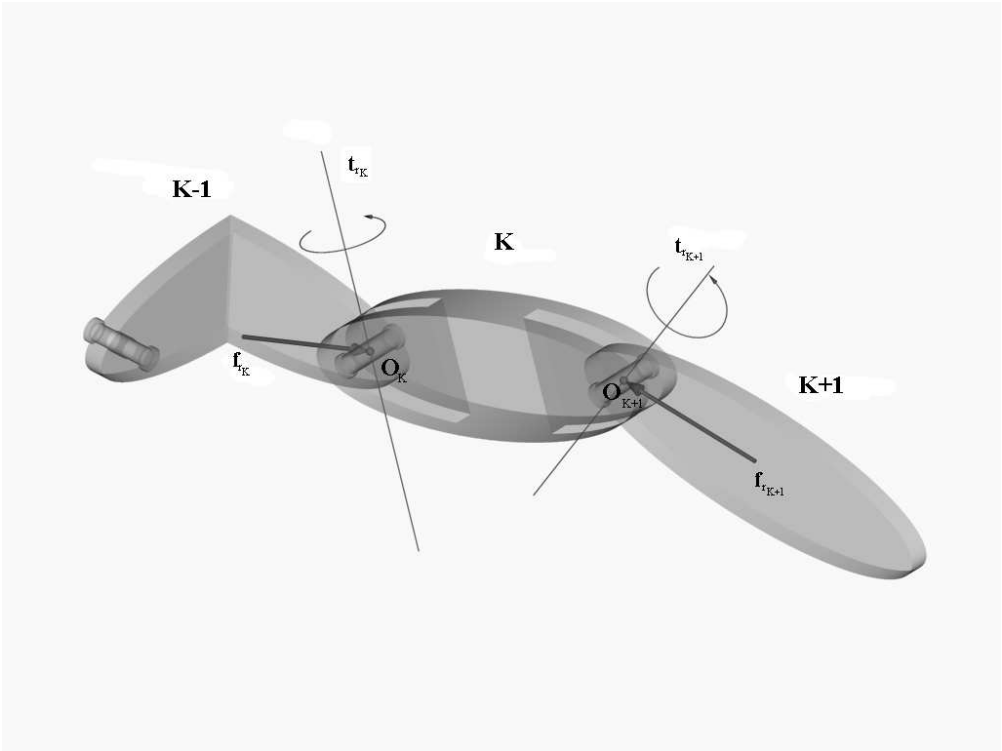


Figure 2: Reactions on body K

By convention, the reactions (torques) from body N are taken with respect to point O_N on the joint axis. To transmit these reactions to origin O_K, the transformation matrix ${}^{K}\mathcal{T}_{N}^{F}$ is used:

$$ {}^{K}\mathcal{T}_{N}^{F} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \widetilde{\mathbf{O_K O_N}} & \mathbf{I} \end{pmatrix} \tag{22} $$

Note that this matrix is constant in the local reference frame. Observe also that the velocities transform in a similar way:

$$ \boldsymbol{\Omega}_N = {}^{N}\mathcal{T}_{K}^{V}\boldsymbol{\Omega}_K + \mathbf{E}_{j_N}\dot{\mathbf{q}}_N + \boldsymbol{\Omega}_{t_N} = \begin{pmatrix} \mathbf{I} & \widetilde{\mathbf{O_N O_K}} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{v}_K \\ \boldsymbol{\omega}_K \end{pmatrix} + \mathbf{E}_{j_N}\dot{\mathbf{q}}_N + \boldsymbol{\Omega}_{t_N} \tag{23} $$

The dimensions of the joint matrix $\mathbf{E}_{j_K}$ $(6 \times n_K)$ and the coordinate velocities vector $\dot{\mathbf{q}}_K$ $(n_K \times 1)$ for a certain joint K is dependent on the number of degrees of freedom n_K of that joint.
The relationship between both transformation matrices is given by:

$$ {}^{K}\mathcal{T}_{N}^{F} = \left({}^{N}\mathcal{T}_{K}^{V}\right)^{T} \tag{24} $$

The projection of the additional reactions on the joint space cannot be ignored, but can be eliminated by means of the equations of motion for body N (20). Grouping the similar terms, remembering to derive with respect to the correct coordinate system using (14) and performing some mathematical manipulations give

$$ \frac{d^K}{dt}\left[\mathbf{P}_K + {}^{K}\mathcal{T}_{N}^{F}\mathbf{P}_N\right] + \boldsymbol{\Omega}_K \times \left[\mathbf{P}_K + {}^{K}\mathcal{T}_{N}^{F}\mathbf{P}_N\right] = \mathbf{T}_K + \mathbf{T}_{r_K} + {}^{K}\mathcal{T}_{N}^{F}\mathbf{T}_N \tag{25} $$

After defining the *articulated* momentum vector $\mathbf{P}^*$ and the *accumulated* force vector $\mathbf{T}^*$ as

$$ \mathbf{P}_K^* = \mathbf{P}_K + {}^{K}\mathcal{T}_{K+1}^{F}\mathbf{P}_{K+1}^* \tag{26} $$

$$ \mathbf{T}_K^* = \mathbf{T}_K + {}^{K}\mathcal{T}_{K+1}^{F}\mathbf{T}_{K+1}^* \tag{27} $$

a concise system of equations is obtained with the same appearance as (20):

$$ \dot{\mathbf{P}}_K^* + \boldsymbol{\Omega}_K \times \mathbf{P}_K^* = \mathbf{T}_K^* + \mathbf{T}_{r_K} \tag{28} $$

5.2 Canonical momenta

Equations (28) are not very useful yet, as unknown reaction forces are still present. To eliminate them, we first need to calculate the canonical momenta of the multibody system, using (19):

$$\mathbf{p}_K = \frac{\partial L}{\partial \dot{\mathbf{q}}_K} = \frac{\partial (\sum_{i=1}^{N} T_i)}{\partial \dot{\mathbf{q}}_K} = \frac{1}{2} \frac{\partial (\sum_{i=1}^{N} \mathbf{\Omega}_i^T \mathbf{M}_i \mathbf{\Omega}_i)}{\partial \dot{\mathbf{q}}_K} \tag{29}$$

T_i being the kinetic energy of body i. In open-loop systems, the spatial velocity $\mathbf{\Omega}_K$ is independent on the outboard joint velocities, and so is the kinetic energy of body K. For the canonical momenta conjugated to body N, we therefore get the same expression as (19):

$$\mathbf{p}_N = \frac{\partial T_N}{\partial \dot{\mathbf{q}}_N} = \mathbf{E}_{j_N}^T \mathbf{P}_N \tag{30}$$

Before deriving the expression for the other bodies, we will first make following observation, which can easily be done by inspecting (23):

$$\frac{\partial \mathbf{\Omega}_L}{\partial \dot{\mathbf{q}}_K} = 0 \qquad \forall K > L \tag{31}$$

$$\frac{\partial \mathbf{\Omega}_L}{\partial \dot{\mathbf{q}}_K} = {}^{L}\mathcal{T}_K^V \mathbf{E}_{j_K} \qquad \forall K \leq L \tag{32}$$

For body K, we get (using (24) and (26))

$$\mathbf{p}_K = \frac{\partial (T_K + ... + T_N)}{\partial \dot{\mathbf{q}}_K} = \frac{\partial \mathbf{\Omega}_K^T}{\partial \dot{\mathbf{q}}_K} \mathbf{P}_K + ... + \frac{\partial \mathbf{\Omega}_N^T}{\partial \dot{\mathbf{q}}_K} \mathbf{P}_N \tag{33}$$

$$= \mathbf{E}_{j_K}^T (\mathbf{P}_K + \sum_{i=K+1}^{N} {}^{K}\mathcal{T}_i^F \mathbf{P}_i) = \mathbf{E}_{j_K}^T \mathbf{P}_K^* \tag{34}$$

Thus, the projection of the articulated momentum vectors on the joint vectors leads to the set of canonical momenta conjugated to the joint coordinates. As was said before, the generalized reaction forces are orthogonal to the space described by the joint matrix ($\mathbf{E}_{j_K}^T \mathbf{T}_{r_K} = \mathbf{0}$). Keeping in mind that $\dot{\mathbf{p}} = \dot{\mathbf{E}}_j^T \mathbf{P} + \mathbf{E}_j^T \dot{\mathbf{P}}$, the projection of equation (28) on $\mathbf{E}_{j_K}$ yields

$$\dot{\mathbf{p}}_K = \mathbf{E}_{j_K}^T \mathbf{T}_K^* - \mathbf{E}_{j_K}^T (\mathbf{\Omega}_K \times \mathbf{P}_K^*) + \dot{\mathbf{E}}_{j_K}^T \mathbf{P}_K^* \tag{35}$$

This is the first set (G_1) of Hamiltonian equations. It can only be evaluated with known values of the velocities, these will be derived in next section.

6 SECOND SET OF EQUATIONS (G_2)

To obtain the second set of Hamiltonian equations, the one involving the function evaluation G_2 (9b), the generalized velocities $\dot{\mathbf{q}}$ need to be expressed as a function of the canonical momenta vector $\mathbf{p}$ and the generalized coordinates $\mathbf{q}$. This can be done starting from the expression of $\mathbf{P}_N$ (11) and writing the spatial velocity as an explicit function of $\dot{\mathbf{q}}_N$ with (23).

$$\mathbf{P}_{N} = \mathbf{M}_{N}\boldsymbol{\Omega}_{N} = \mathbf{M}_{N}({}^{N}\mathcal{T}^{V}_{N-1}\boldsymbol{\Omega}_{N-1} + \mathbf{E}_{j_N}\dot{\mathbf{q}}_N + \boldsymbol{\Omega}_{t_N}) \tag{36}$$

Projecting these equations on the joint space and rearranging the terms give an expression for the generalized coordinates:

$$\dot{\mathbf{q}}_N = \mathbf{M}^{-1}_{j_N}[\mathbf{p}_N - \mathbf{E}^{T}_{j_N}\mathbf{M}_{N}({}^{N}\mathcal{T}^{V}_{N-1}\boldsymbol{\Omega}_{N-1} + \boldsymbol{\Omega}_{t_N})] \tag{37}$$

with $\mathbf{M}_j$ the joint mass matrix defined as

$$\mathbf{M}_j = \mathbf{E}^{T}_{j}\mathbf{M}\mathbf{E}_j \tag{38}$$

The expression is of the required form, as the velocity vector $\boldsymbol{\Omega}$ is dependent on the joint velocities of all inboard links. If similar equations are found for all other bodies, the velocities can be computed recursively starting from the base. These equations can be obtained by first eliminating $\dot{\mathbf{q}}_N$ from (36) by means of (37), and rearranging the terms:

$$\mathbf{P}_{N} = \mathbf{M}'_{N}({}^{N}\mathcal{T}^{V}_{N-1}\boldsymbol{\Omega}_{N-1} + \boldsymbol{\Omega}_{t_N}) + \mathbf{D}'_{N} \tag{39}$$

with

$$\mathbf{M}'_{N} = \mathbf{M}_{N} - \mathbf{M}_{N}\mathbf{E}_{j_N}\mathbf{M}^{-1}_{j_N}\mathbf{E}^{T}_{j_N}\mathbf{M}_{N} \tag{40}$$

$$\mathbf{D}'_{N} = \mathbf{M}_{N}\mathbf{E}_{j_N}\mathbf{M}^{-1}_{j_N}\mathbf{p}_{N} \tag{41}$$

$\mathbf{M}'$ is called the reduced mass matrix. $\mathbf{D}'$ is a remainder term. Substitution of (39) in (26) results in the desired formulation.

$$\mathbf{P}^{*}_{K} = \mathbf{M}^{*}_{K}\boldsymbol{\Omega}_{K} + \mathbf{D}_{K} \tag{42}$$

with

$$\mathbf{M}^{*}_{K} = \mathbf{M}_{K} + {}^{K}\mathcal{T}^{F}_{N}\mathbf{M}'_{N}\,{}^{N}\mathcal{T}^{V}_{K} \tag{43}$$

$$\mathbf{D}_{K} = {}^{K}\mathcal{T}^{F}_{N}(\mathbf{M}'_{N}\boldsymbol{\Omega}_{t_N} + \mathbf{D}'_{N}) \tag{44}$$

$\mathbf{M}^*$ is the articulated mass matrix and $\mathbf{D}$ the momentum remainder term. We denote the projection of the remainder momentum vector on the joint space with the vector $\mathbf{d}$. $\mathbf{P}^*_K$ does now have a form similar to $\mathbf{P}_N$ and the velocities $\dot{\mathbf{q}}_K$ can be found, as for body N:

$$\dot{\mathbf{q}}_K = \mathbf{M}^{-1}_{j_K}[(\mathbf{p}_K - \mathbf{d}_K) - \mathbf{E}^T_{j_K}\mathbf{M}^*_K({}^K\mathcal{T}^V_{K-1}\boldsymbol{\Omega}_{K-1} + \boldsymbol{\Omega}_{t_K})] \tag{45}$$

Repeating the procedure for the other bodies reveals a slightly different structure for $\mathbf{M}_{j_K}$, $\mathbf{M}'_K$ and $\mathbf{D}'_K$:

$$\mathbf{M}_{j_K} = \mathbf{E}^T_{j_K}\mathbf{M}^*_K\mathbf{E}_{j_K} \tag{46}$$

$$\mathbf{M}'_K = \mathbf{M}^*_K - \mathbf{M}^*_K\mathbf{E}_{j_K}\mathbf{M}^{-1}_{j_K}\mathbf{E}^T_{j_K}\mathbf{M}^*_K \tag{47}$$

$$\mathbf{D}'_K = \mathbf{M}^*_K\mathbf{E}_{j_K}\mathbf{M}^{-1}_{j_K}(\mathbf{p}_K - \mathbf{d}_K) + \mathbf{D}_K \tag{48}$$

The spatial velocity $\boldsymbol{\Omega}_0$ of the base is known, so the vector of coordinate velocities $\dot{\mathbf{q}}_1$ at joint 1 can be computed. These can thereafter be used to compute the velocity vector $\boldsymbol{\Omega}_2$ which on its turn enable the calculation of the joint velocities $\dot{\mathbf{q}}_2$ and so on. All joint velocities can be found by forward recursion. The obtained spatial velocity vectors are used to compute the first set of Hamiltonian equations (35) derived in the previous section. So, in a first, backward recursion, the articulated mass matrices $\mathbf{M}^*$, the momentum remainder vectors $\mathbf{D}$ and the accumulated forces $\mathbf{T}$ are calculated. In a subsequent, forward recursion, the joint velocities $\dot{\mathbf{q}}$ and time derivatives of the canonical momenta $\dot{\mathbf{p}}$ are computed. Acceleration based algorithms typically need a third recursion step for the forward kinematics. This gives an additional advantage to the canonical momenta based method, when implemented on a parallel computing architecture.

In the case only pin-joints are used, a thorough inspection of the algorithm revealed a maximum of 363 operations are needed for each body. Due to simplifications at the first and last bodies, this amount is reduced with at least 475 operations for the complete mechanism. This can be written: $363n - 475$, with n the number of bodies (degrees of freedom). This formula is applicable for $n \geq 3$. For comparison, a list of acceleration based algorithms and their amount of operations is shown in following tabel.

7 CONCLUSIONS

In this paper, a previously introduced recursive $O(n)$ algorithm for the derivation of a set of Hamiltonian equations has been generalized to cope with any kind of holonomic joint. The method is very promising compared to acceleration based algorithms thanks to a reduced number of arithmetical operations needed to obtain the equations of motion, a potentially advantageous behavior during numerical integration and a reduced number of recursion steps.

Algorithm	Add.	Mult.	Total
Featherstone[10]	$275n - 18$	$336n - 220$	$611n - 238$
Vukobratović[14]	$231n - 294$	$249n - 272$	$480n - 566$
Valášek[21]	$206n - 345$	$226n - 343$	$432n - 688$
Rein[13]	$195n - 247$	$216n - 317$	$411n - 669$
Naudet-Lefeber[18]	$178n - 230$	$185n - 245$	$363n - 475$

Table 1: Number of required operations

8 FUTURE WORK

The next step in this research will be to develop a method to obtain the reaction forces and torques between the bodies of an open-loop multibody system. Afterwards, the algorithm will be extended to handle systems with kinematical loops. Preliminary research pointed out that this should be possible with a recursive method.

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